Quiz 4 – Additional Problems

① Solve $\quad y(x+y+1)\,dx + (x+2y)\,dy = 0$

(1)(a) shows this is not exact
(1)(b) we find the integrating factor $I.F. = e^x$

SOL:
multiply by I.F., e^x :

$$\underbrace{e^x y(x+y+1)}_{M(x,y)}\,dx + \underbrace{e^x(x+2y)}_{N(x,y)}\,dy = 0$$

$$f(x,y) = \int M(x,y)\,dx + g(y)$$

$$= \int e^x y(x+y+1)\,dx + g(y)$$

$$= \int (e^x yx + e^x y^2 + e^x y)\,dx + g(y)$$

$$= ye^x(x-1) + e^x y^2 + e^x y + g(y) \quad \cancel{yxe^x + e^x y^2 + g(y)}$$

$$= yxe^x + e^x y^2 + g(y)$$

$$\frac{\partial f}{\partial y} = e^x(x-1) + 2e^x y + e^x + g'(y) = e^x(x+2y)$$

$$\cancel{xe^x} \cancel{-e^x} + \cancel{2e^x y} \cancel{+e^x} + g'(y) = \cancel{xe^x} + \cancel{2ye^x}$$

$$g'(y) = 0$$

$$g(y) = C$$

$$f(x,y) = ye^x(x-1) + e^x y^2 + e^x y + C$$

$$= yxe^x + y^2e^x + C$$

solution : $f(x,y) = D$

$$yxe^x + y^2e^x + C = D$$

$$\boxed{xye^x + y^2e^x = E}$$

② Solve: $(y^2 + yx)dx + x^2 dy = 0$

(2)(a) shows this is homogeneous
(2)(b) gives a possible substitution.

SOL: Let's use the substitution $u = \frac{y}{x}$ (you can also use $u = x/y$)

So $y = ux$

$dy = u\,dx + x\,du$

substituting into

$$(y^2 + yx)dx + x^2 dy = 0$$

we get

$$((ux)^2 + (ux)x)\,dx + x^2(u\,dx + x\,du) = 0$$

$$(u^2x^2 + ux^2)dx + x^2u\,dx + x^3du = 0$$

$$u^2x^2dx + ux^2dx + x^2u\,dx + x^3du = 0$$

$$u^2x^2dx + 2ux^2dx + x^3du = 0$$

$$(u^2x^2 + 2ux^2)\,dx = -x^3du$$

$$x^2(u^2 + 2u)\,dx = -x^3du$$

$$\frac{x^2}{x^3}dx = \frac{-1}{u^2+2u}du$$

$$\frac{1}{x}dx = -\frac{1}{u(u+2)}du = \left[\frac{A}{u} + \frac{B}{u+2}\right]du$$

$$A(u+2) + Bu = -1$$

$$Au + 2A + Bu = -1$$

$$(A+B)u + 2A = -1$$

$$A = -\frac{1}{2}$$

$$A+B = 0 \Rightarrow B = \frac{1}{2}$$

$$\int \frac{1}{x}\,dx = \int \left[\frac{-1}{2} \cdot \frac{1}{u} + \frac{1}{2} \cdot \frac{1}{(u+2)} \right] du$$

$$\ln|x| + C_1 = -\frac{1}{2} \ln|u| + \frac{1}{2} \ln|u+2| + C_2$$

$$\ln|x| + \frac{1}{2} \ln|u| - \frac{1}{2} \ln|u+2| = C$$

$$\cancel{\ln|x| + \ln\left(|u|^{1/2}\right) - \ln\left(|u+2|^{1/2}\right) = C}$$

$$\cancel{\ln\left[\frac{x\sqrt{|u|}}{\sqrt{u+2}}\right] = C}$$

$$2\ln|x| + \ln|u| - \ln|u+2| = D$$

$$\ln(x^2) + \ln|u| - \ln|u+2| = D$$

$$\ln\left(\frac{x^2|u|}{|u+2|}\right) = D$$

$$\frac{x^2|u|}{|u+2|} = e^D = E$$

$$\frac{x^2 u}{u+2} = F$$

$$\frac{x^2 (y/x)}{y/x + 2} = F$$

$$x^2\left(\frac{y}{x}\right) = F\left(\frac{y}{x} + 2\right)$$

$$\boxed{x^2 y = F(y + 2x)}$$