Quiz # 10 - HW Quiz

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Solve $x^2y'' + xy' - y = \ln x$

by variation of parameters.

SOLUTION: This is 2nd order linear, Cauch-Euler Equation, Nonhomogeneous.

Step 1: ~~[illegible]~~

Find y_c the general solution of the homogeneous D.E.:

$$x^2y'' + xy' - y = 0$$

Aux. Eqn.:

$$am^2 + (b-a)m + c = 0$$

$$1m^2 + (1-1)m + (-1) = 0$$

$$m^2 + 0m - 1 = 0$$

$$m^2 - 1 = 0$$

$$(m-1)(m+1) = 0$$

$$m = 1 \quad \Big| \quad m = -1$$

← 2 distinct roots... so..

$$y_c = \underbrace{C_1 x^1}_{y_1} + \underbrace{C_2 x^{-1}}_{y_2}$$

Step 2: Find y_p of $x^2y'' + xy' - y = \ln x$

using variation of parameters.

(i) Standard Form: $\frac{x^2y''}{x^2} + \frac{xy'}{x^2} - \frac{y}{x^2} = \frac{\ln(x)}{x^2}$

$$y'' + \frac{1}{x}y' - \frac{y}{x^2} = \underbrace{\frac{\ln x}{x^2}}_{f(x)}$$

(ii) $y_1 = x$, $y_2 = x^{-1}$, $f(x) = \frac{\ln x}{x^2}$

↓

$$W = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^{-1} \\ 1 & -1x^{-2} \end{vmatrix} = x(-1x^{-2}) - (1)(x^{-1})$$

$$= -x^{-1} - x^{-1}$$

$$= -2x^{-1}$$

$$= \frac{-2}{x}$$

so

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{-(x^{-1})\left(\frac{\ln(x)}{x^2}\right)}{\frac{-2}{x}} = \frac{\left(\frac{1}{x}\right)\left(\frac{\ln(x)}{x^2}\right)}{\left(\frac{2}{x}\right)} = \frac{\ln(x)}{2x^2}$$

$$u_1 = \int \frac{\ln(x)}{2x^2}\, dx$$ ← Integration by parts.

$$u = \ln(x) \qquad dv = \frac{1}{2x^2} = \frac{1}{2}x^{-2}$$

$$du = \frac{1}{x}\,dx \qquad v = \frac{1}{2}\frac{x^{-1}}{-1} = -\frac{1}{2x}$$

$$= -\frac{1}{2x}\ln(x) - \int \left(-\frac{1}{2x}\right)\left(\frac{1}{x}\right)dx$$

$$= \frac{-1}{2x}\ln(x) + \frac{1}{2}\int \frac{1}{x^2}\,dx$$

$$= \frac{-1}{2x}\ln(x) + \frac{1}{2}\int x^{-2}\,dx$$

$$= \frac{-1}{2x}\ln(x) + \frac{1}{2}\frac{x^{-1}}{-1}$$

$$= -\frac{1}{2x}\ln(x) - \frac{1}{2x}$$

$$= \frac{-\ln(x) - 1}{2x}$$

↓

$$u_2' = \frac{y_1 f(x)}{W} = \frac{x\left(\frac{\ln x}{x^2}\right)}{\left(\frac{-2}{x}\right)}$$

$$= \frac{-\ln x}{2}$$ Integration by parts.

$$u_2 = \int \frac{-\ln x}{2}\,dx = -\frac{1}{2}\int \ln(x)\,dx$$

$u = \ln(x)$ $\quad dv = 1\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = x$

$$= -\frac{1}{2}\left(x\ln x - \int x\left(\tfrac{1}{x}\right)dx\right)$$

$$= -\frac{1}{2}\left(x\ln x - \int 1\,dx\right)$$

$$= -\frac{1}{2}x\ln x + \frac{1}{2}x$$

$$= -\frac{1}{2}\left(x\ln x - x\right)$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{-\ln x - 1}{2x}\right)(x) + \left(-\tfrac{1}{2}(x\ln x - x)\right)(x^{-1})$$

$$= \frac{-\ln x - 1}{2} + \frac{-1}{2}\left(\frac{\cancel{x}\ln x - \cancel{x}}{\cancel{x}}\right)$$

$$= \frac{-\ln x - 1}{2} + -\frac{1}{2}(\ln x - 1)$$

$$= \frac{-\ln x - 1 - \ln x + 1}{2}$$

$$= \frac{-2\ln x}{2} = -\ln x$$

Step3 general solution $y = y_c + y_p = \boxed{c_1 x + c_2 x^{-1} + (-\ln x)}$