Math 371 ODEs: Midterm Review

Topics Covered in Midterm: First-Order Differential Equations

- Linear Equations: Method of Integrating Factors
- Separable Equations
- Exactness
- Homogeneous Equations
- Bernoulli Equations

Topics Covered in Midterm: Second-Order Differential Equations

- Reduction of Order
- Homogeneous Linear Equations with Constant Coefficients
- Nonhomogeneous Linear Equations with Constant Coefficients: Undtermined Coefficients
- Nonhomogeneous Linear Equations: Variation of Parameters
- Cauchy-Euler Equations

Part I First-Order ODEs

Linear Equations

To find a solution of equation $\frac{dy}{dx} + P(x)y = f(x)$, do the following

Step 1. Calculate the integrating factor: $\rho(x) = e^{\int P(x)dx}$

Step 2. Multiply both sides of the DE by $\rho(x)$:

$$\rho(x)\frac{dy}{dx} + \rho(x)P(x)y = \rho(x)f(x)$$

Step 3. Write the LHS of the resulting equation in Step 2 as

$$\rho(x)\frac{dy}{dx} + \rho(x)P(x)y = D_x[\rho(x)y(x)]$$

Step 4. Integrate the equation $D_x[\rho(x)y(x)] = \rho(x)f(x)$:

$$\rho(x)y(x) = \int \rho(x)f(x)dx + C$$

Step 5. Solve for y:

$$y(x) = \frac{1}{\rho(x)}\int \rho(x)f(x)dx + \frac{C}{\rho(x)}.$$

Linear Equations: Things to be Remembered

The recipe only works for the linear equations in the standard form. Thus if you need to solve equation

$$a_1(x)y' + a_0(x)y = f(x)$$

you should turn it into the standard form

$$y' + \frac{a_0(x)}{a_1(x)}y = \frac{f(x)}{a_1(x)}$$

before using the method of integrating factors.

Separable Equations

To solve an separable equation $\frac{dy}{dx} = H(x, y)$, do the following

Step 1. Write the DE in the separable form:

$$\frac{dy}{dx} = H(x, y) = g(x)h(y) \Rightarrow \frac{dy}{h(y)} = g(x)dx$$

Step 2. Integrate on both sides to get an implicit solution:

$$\int \frac{dy}{h(y)} = \int g(x)dx + C$$

Separable Equations: Things to be Remembered

The recipe may not be able to find the singular solutions, that is, those trivial solutions $y = y_0$ such that $h(y_0) = 0$. Thus to get all solutions, you should FIRST check the singular solutions, then use the above method to get an implicit solution.

Exactness

If DE $Mdx + Ndy = 0$ satisfies

$$M_y = N_x$$

then we could find a general solution by doing the following

Step 1. Integrate $F_x = M$ w.r.t. x.

Step 2. Integrate $F_y = N$ w.r.t. y.

Step 3. F can be expressed as a union of the terms obtained in Steps 2 and 3.

Exactness: General Case

To solve a nonexact DE $M(x,y)dx + N(x,y)dy = 0$, we may do the following

Step 1. Multiply the DE by a factor $\mu(x,y)$: $\mu M dx + \mu N dy = 0$

Step 2. Apply the Criterion for Exactness to the new DE:

$$\frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N] \Rightarrow M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y} = N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x}$$

Step 3. Obtain an ODE in μ and solve for μ:

- Drop the term of μ_x : $\dfrac{d\mu}{dy} = \dfrac{N_x - M_y}{M}\mu \Rightarrow \mu = e^{\int \frac{N_x - M_y}{M} dy}$
- Drop the term of μ_y : $\dfrac{d\mu}{dx} = \dfrac{M_y - N_x}{N}\mu \Rightarrow \mu = e^{\int \frac{M_y - N_x}{N} dx}$

Step 4. Solve the new DE obtained in Step 1 by the method of exact equations.

To solve homogeneous equation $y' = f(y/x)$, do the following

Step 1. Transform the original equation into a separable equation via the substitution $v = \frac{y}{x}$:

$$x\frac{dv}{dx} = f(v) - v.$$

Step 2. Solve the separable equation obtained in Step 1.

Step 3. Substitute v back to get the general solution:

To solve Bernoulli equation $\frac{dy}{dx} + P(x)y = Q(x)y^n$, do the following

Step 1. Transform the original equation into a linear equation via the substitution $v = y^{1-n}$:

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

Step 2. Use the method of integrating factors to solve the resulting linear DE.

Step 3. Substitute v back to obtain the general solution.

Part II Second-Order ODEs

Reduction of Order

If y_1 is a solution of DE

$$y'' + P(x)y' + Q(x)y = 0,$$

then we a second linearly independent solution y_2 can be obtained by the following formula

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

Reduction of Order: Things to be Remembered

The formula in reduction of order only works for the equations in the standard form. Thus if you need to find a second solution of

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

you should turn it into the standard form

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = 0$$

before using the formula.

Homogeneous Linear Second-Order Equations with Constant Coefficients

To solve equation $ay'' + by' + cy = 0$, where a, b, and c are constants, do the following

Step 1. Find the corresponding characteristic equation:

$$am^2 + bm + c = 0$$

Step 2. Solve this characteristic equation:

$$m_1, m_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Step 3. Get the general solution:

Case 1. $m_1 \neq m_2$ real: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case 2. $m_1 = m_2$ real: $y = (c_1 + c_2 x) e^{m_1 x}$

Case 3. $m_1, m_2 = \alpha \pm i\beta$: $y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$

Homogeneous Linear Equations with Constant Coefficients: General Case

To solve equation $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$, where a_i, $i = 0, 1, \cdots, n$, are constants, do the following
Step 1. Find the corresponding characteristic equation:

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0$$

Step 2. Solve this characteristic equation.

Step 3. Suppose that m is a root with multiplicity k. Then the part of a general solution corresponding to m is of the form:

Case 1. m is real: $(c_0 + c_1 x + \cdots + c_{k-1} x^{k-1}) e^{mx}$

Case 2. $m = \alpha \pm i\beta$: $\left(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}\right) e^{\alpha x} \cos(\beta x) + \left(d_1 + d_2 x + d_3 x^2 + \cdots + d_k x^{k-1}\right) e^{\alpha x} \sin(\beta x)$

Undtermined Coefficients

Consider the DE $ay'' + by' + cy = g(x)$ with $g(x)$ of either form $P_l(x)e^{mx}\cos(kx)$ or $P_l(x)e^{mx}\sin(kx)$, where $P_l(x)$ is a polynomial in x of degree l. To find a particular solution y_p, do the following

Step 1. Find its complementary function y_c.

Step 2. Take the trial function as

$$\begin{aligned} y_p(x) =& (A_0 + A_1 x + A_2 x^2 + \cdots + A_l x^l)e^{mx}\cos(kx) \\ &+ (B_0 + B_1 x + B_2 x^2 + \cdots + B_l x^l)e^{mx}\cos(kx). \end{aligned}$$

Step 3. Check the duplicate terms.

Step 4. Eliminate the duplicate part by multiplying by x^s, where $s > 0$ is the smallest such integer.

Variation of Parameters

To find a particular solution of the nonhomogeneous linear DE $y'' + a_1(x)y' + a_0(x)y = g(x)$, do the following

Step 1. Find the complementary function $y_c = c_1 y_1 + c_2 y_2$.

Step 2. Assume $y_p = u_1 y_1 + u_2 y_2$.

Step 3. Write down the system of equations of u_1', u_2':

$$u_1' y_1 + u_2' y_2 = 0, \quad u_1' y_1' + u_2' y_2' = g(x)$$

Step 4. Solve for u_1', u_2':

$$u_1' = -\frac{y_2 g(x)}{W(y_1, y_2)} \quad \text{and} \quad u_2' = \frac{y_1 g(x)}{W(y_1, y_2)},$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2.

Step 5. Obtain u_1 and u_2:

$$u_1 = -\int \frac{y_2 g(x)}{W(y_1, y_2)} dx \quad \text{and} \quad u_2 = \int \frac{y_1 g(x)}{W(y_1, y_2)} dx$$

Variation of Parameters: Things to be Remebered

The formulas in variation of parameters only work for the equations in the standard form. Thus if you need to find a particular solution of

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

you should turn it into the standard form

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \frac{g(x)}{a_2(x)}.$$

Then in Step 3 the system of equations of u_1', u_2' is:

$$u_1'y_1 + u_2'y_2 = 0, \quad u_1'y_1' + u_2'y_2' = \frac{g(x)}{a_2(x)}$$

Cauchy-Euler Equations

To solve equation $ax^2y'' + bxy' + cy = 0$, where a, b, and c are constants, do the following

Step 1. Find the corresponding characteristic equation:

$$am(m-1) + bm + c = 0$$

Step 2. Solve this characteristic equation:

$$m_1, m_2 = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$

Step 3. Get the general solution:

Case 1. $m_1 \neq m_2$ real: $y = c_1x^{m_1} + c_2x^{m_2}$

Case 2. $m_1 = m_2$ real: $y = (c_1 + c_2 \ln x)x^{m_1}$

Case 3. $m_1, m_2 = \alpha \pm i\beta$: $y = c_1x^{\alpha}\cos(\beta \ln x) + c_2x^{\alpha}\sin(\beta \ln x)$

Cauchy-Euler Equations: Things to be Remebered

• For the Cauchy-Euler equation $ax^2y'' + bxy' + cy = 0$, if we introduce the substitution $x = e^t$, then the equation can be rewritten as

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0,$$

which is a linear equation with constant coefficients.

• To solve the nonhomogeneous equation $ax^2y'' + bxy' + cy = g(x)$, we may use the variation of parameters. Notice that we need FRIST turn the DE into its standard form so that the system of equations of u_1', u_2' is:

$$u_1'y_1 + u_2'y_2 = 0, \quad u_1'y_1' + u_2'y_2' = \frac{g(x)}{ax^2}$$