Math 371 - Midterm Review Questions

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Problem 1

$$y' = e^{3x+2y}.$$

Seperable (Section 2.2)

$$\frac{dy}{dx} = e^{3x+2y}$$
$$\frac{dy}{dx} = e^{3x}e^{2y}$$
$$\frac{dy}{e^{2y}} = e^{3x}dx$$
$$\int e^{-2y}dy = \int e^{3x}dx$$
$$-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + C.$$

Any singular solutions? NO! $e^{2y} \neq 0$ for all $y \in \mathbb{R}$.

Problem 2

$$\frac{dy}{dx} = \frac{x^2 y}{x^3+1}.$$

Seperable (Section 2.2)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x^2 y}{x^3+1} \\
\frac{dy}{y} &= \frac{x^2}{x^3+1}dx \\
\int \frac{dy}{y} &= \int \frac{x^2}{x^3+1}dx \\
\ln|y| &= \frac{1}{3}\ln|x^3+1| + C \text{ (u-substitution).}
\end{aligned}$$

Any singular solutions? YES! $y = 0$.

Problem 3

$$x\frac{dy}{dx} - y = x^2 \sin x.$$

1st Order Linear (Section 2.3)

1. Standard Form: $\dfrac{dy}{dx} - \dfrac{y}{x} = x \sin x.$
2. Integrating Factor:

$$\begin{aligned} e^{\int (-1/x)dx} = e^{-\ln|x|} &= e^{\ln(|x|^{-1})} \\ &= |x|^{-1} \\ &= 1/|x| = 1/x \text{ if } x > 0. \end{aligned}$$

3. DE becomes:

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = \sin x$$

$$\frac{1}{x}y = \int \sin x dx = -\cos x + C$$

$$y = -x\cos x + Cx.$$

Problem 4 - Exact Equations (Section 2.4)

Determine whether this DE is exact. If it is solve it,

$$(2x + y)dx + (x + 6y)dy = 0.$$

$M(x, y) = 2x + y$ and $N(x, y) = x + 6y$.

(1) Check $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$:

- $\frac{\partial M}{\partial y} = 1$.
- $\frac{\partial N}{\partial x} = 1$.

Since $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$, the DE is exact!

(2)

$$\begin{aligned} f(x, y) &= \int M(x, y)dx + g(y) \\ &= \int (2x + y)dx + g(y) \\ &= x^2 + yx + g(y). \end{aligned}$$

(3) Set $\frac{\partial}{\partial y} f(x,y) = N(x,y)$:

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\left(x^2 + yx + g(y)\right)$$
$$= 0 + x + g'(y) = N(x,y) = x + 6y.$$

So, $g'(y) = 6y$.

(4) Solve for $g(y)$ and substitute into $f(x,y)$ from step (2):

$$g(y) = \int 6y dy = 3y^2.$$

So,

$$f(x,y) = x^2 + yx + 3y^2.$$

(5) Solution:

$$f(x,y) = C.$$

Hence,

$$x^2 + yx + 3y^2 = C.$$

Problem 5

$$ydx - (2x+2y)dy = 0.$$

Homogeneous (of degree) Section 2.5

1. Substitute $y = ux$ (or $x = uy$). Also, $dy = udx + xdu$.
2. DE becomes:

$$\begin{aligned}
uxdx - (2x+2ux)(udx+xdu) &= 0\\
uxdx - (2xudx + 2x^2du + 2u^2xdx + 2ux^2du) &= 0\\
uxdx - 2xudx - 2x^2du - 2u^2xdx - 2ux^2du &= 0\\
(ux - 2xu - 2u^2x)dx + (-2x^2 - 2ux^2)du &= 0\\
x(-u-2u^2)dx + x^2(-2-2u)du &= 0\\
x(-u-2u^2)dx &= -x^2(-2-2u)du\\
\frac{x}{x^2}dx &= \frac{-(-2-2u)}{-u-2u^2}du\\
\frac{x}{x^2}dx &= \frac{2(1+u)}{-u(1+2u)}du
\end{aligned}$$

Problem 5 - Faster!

$$ydx - (2x + 2y)dy = 0.$$

Homogeneous (of degree) Section 2.5

1. Substitute $x = uy$. Also, $dx = udy + ydu$.
2. DE becomes:

$$\begin{aligned}
y(udy + ydu) - (2uy + 2y)dy &= 0 \\
yudy + y^2du - 2uydy - 2ydy &= 0 \\
(yu - 2uy - 2y)dy + y^2du &= 0 \\
(-uy - 2y)dy + y^2du &= 0 \\
y(-u - 2)dy + y^2du &= 0 \\
y(-u - 2)dy &= -y^2du \\
\frac{y}{y^2}dy &= \frac{1}{u + 2}du \\
\ln|y| + C &= \ln|u + 2| \\
\ln|y| + C &= \ln|x/y + 2|
\end{aligned}$$

Problem 6

$$x\frac{dy}{dx} + y = x^2y^2.$$

Bernoulli's Equation(Section 2.5)

1. Substitute $u = y^{1-n} = y^{1-2} = y^{-1}$, i.e., $y = 1/u$, and $\frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}$
2. DE becomes:

$$x\left(-\frac{1}{u^2}\frac{du}{dx}\right) + \frac{1}{u} = x^2\left(\frac{1}{u}\right)^2$$
$$\frac{du}{dx} - \frac{1}{x}u = -x$$

Now this is 1st Order Linear. Solution:

$$y = 1/(-x^2 + Cx).$$

Problem 7 - Reduction of Order (Section 3.2)

$y_1 = e^{x/3}$ is a solution of $6y'' + y' - y = 0$. Use reduction of order to find a second independent solution y_2.

1. Standard Form: $y'' + P(x)y' + Q(x)y = 0$.
   standard form here is

$$y'' + (1/6)y' - (1/6)y = 0.$$

   So $P(x) = 1/6$.

2. Use the formula

$$\boxed{y_2 = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx.}$$

$$y_2 = e^{x/3} \int \frac{e^{-\int (1/6)dx}}{(e^{x/3})^2} dx$$

$$= e^{x/3} \int \frac{e^{-x/6}}{e^{2x/3}} dx = e^{x/3} \int e^{-5x/6} dx = -\frac{6}{5} e^{-x/2}$$

Problem 8

$$y'' - 10y' + 25y = 0$$

2nd Order Linear Homogeneous with Constant Coef. (Section 3.3)

1. Solve auxiliary equation:

   $$1m^2 - 10m + 25 = 0 \implies (m-5)(m-5) = 0 \implies m = 5, m = 5.$$

   Repeated roots, so...

2. General solution:

   $$y = C_1 e^{5x} + C_2 x e^{5x}.$$

Problem 9

$$y''' - 10y'' + 25y' = 0$$

3nd Order Linear Homogeneous with Constant Coef. (Section 3.3)

1. Solve auxiliary equation:

$$1m^3 - 10m^2 + 25m = 0$$
$$m(m-5)(m-5) = 0 \implies m = 0, m = 5, m = 5.$$

   One distinct root and one repeated roots, so...

2. General solution:

$$y = C_1 e^{0x} + C_2 e^{5x} + C_3 x e^{5x}.$$

$$y = C_1 + C_2 e^{5x} + C_3 x e^{5x}.$$

Problem 10

$$y'' - 10y' + 25y = e^x$$

2nd Order Linear Nonomogeneous with Constant Coef. (Section 3.4/3.5)

(1) FIND y_c, the general solution to the associated homogeneous DE:

$$y'' - 10y' + 25y = 0.$$

We just did this one (Problem 8) Solve auxiliary equation:

$1m^2 - 10m + 25 = 0 \implies (m-5)(m-5) = 0 \implies m = 5, m = 5.$

Repeated roots, so...

$$y_c = C_1 e^{5x} + C_2 x e^{5x}.$$

(2) FIND y_p, a particular solution of $y'' - 10y' + 25y = e^x$

FIND y_p, a particular solution of $y'' - 10y' + 25y = e^x$.
Here, $g(x) = e^x$.
2 OPTIONS HERE:

1. Undetermined Coefficients (Section 3.4) Because $g(x)$ is a combination of polynomials, exponentials, sines and cosines.
2. Variation of Parameters (Section 3.5)

Which option do we use?
Let's try Option 1: Undetermined Coefficients:
FORM of y_p from $g(x)$:

$$y_p = Ae^x.$$

This does not need to be adjusted, since $y_1 = e^{5x}$ and $y_2 = xe^{5x}$ do not appear in this form.
So this seems like a good choice!

$$y_p = Ae^x$$
$$y_p' = Ae^x$$
$$y_p'' = Ae^x$$

Substituting into $y'' - 10y' + 25y = e^x$:

$$\begin{aligned} Ae^x - 10Ae^x + 25Ae^x &= e^x \\ (A - 10A + 25A)e^x &= e^x \\ (16A)e^x &= e^x \end{aligned}$$

So $16A = 1 \implies A = 1/16$.
Hence,

$$y_p = \frac{1}{16}e^x.$$

(3) The general solution of $y'' - 10y' + 25y = e^x$ is

$$y = y_c + y_p = C_1e^{5x} + C_2xe^{5x} + \frac{1}{16}e^x.$$

Problem 11

$$y'' - 10y' + 25y = e^{5x}$$

2nd Order Linear Nonomogeneous with Constant Coef. (Section 3.4/3.5)

(1) FIND y_c, the general solution to the associated homogeneous DE:

$$y'' - 10y' + 25y = 0.$$

We just did this one (Problem 8) Solve auxiliary equation:

$1m^2 - 10m + 25 = 0 \implies (m-5)(m-5) = 0 \implies m = 5, m = 5.$

Repeated roots, so...

$$y_c = C_1 e^{5x} + C_2 x e^{5x}.$$

(2) FIND y_p, a particular solution of $y'' - 10y' + 25y = e^{5x}$

FIND y_p, a particular solution of $y'' - 10y' + 25y = e^{5x}$.
Here, $g(x) = e^{5x}$.
2 OPTIONS HERE:

1. Undetermined Coefficients (Section 3.4) Because $g(x)$ is a combination of polynomials, exponentials, sines and cosines.
2. Variation of Parameters (Section 3.5)

Which option do we use? Let's try Option 1: Undetermined Coefficients: FORM of y_p from $g(x)$:

$$y_p = Ae^{5x}.$$

This needs to be adjusted! since $y_1 = e^{5x}$ is in y_p multiply by x:

$$y_p = Axe^{5x}.$$

Again this needs to be adjusted! since $y_2 = xe^{5x}$ is also in y_p multiply by another x:

$$y_p = Ax^2e^{5x}.$$

So this seems like a BAD choice! Let's use option 2.

FIND y_p, a particular solution of $y'' - 10y' + 25y = e^{5x}$ using variation of paramters.

1. Standard Form: $y'' - 10y' + 25y = e^{5x}$ DONE!
2. $y_1 = e^{5x}$, $y_2 = xe^{5x}$, $f(x) = e^{5x}$ and the Wronskian of y_1 and y_2 is $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{5x} & xe^{5x} \\ 5e^{5x} & 5xe^{5x} + e^{5x} \end{vmatrix} = e^{10x}$.
3. $u_1' = \dfrac{-y_2 f(x)}{W} = \dfrac{-xe^{5x}e^{5x}}{e^{10x}} = -x$ so $u_1 = -x^2/2$.

   $u_2' = \dfrac{y_1 f(x)}{W} = \dfrac{e^{5x}e^{5x}}{e^{10x}} = 1$ so $u_2 = x$.
4. Hence, $y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{2}x^2 e^{5x} + x x e^{5x} = \frac{1}{2}x^2 e^{5x}$.

(3) The general solution of $y'' - 10y' + 25y = e^{5x}$ is

$$y = y_c + y_p = C_1 e^{5x} + C_2 x e^{5x} + \frac{1}{2}x^2 e^{5x}.$$

Problem 12

$$x^2y'' - 2xy' - 4y = 0.$$

2nd Order Homogeneous Cauchy-Euler Equation (Section 3.6)

- Solve auxiliary equation: $1m^2 + (-2-1)m - 4 = 0 \implies m^2 - 3m - 4 = 0 \implies (m-4)(m+1) = 0$. So $m = 4, m = -1$.Distinct roots...
- General Solution:

$$y = C_1x^4 + C_2x^{-1}$$