Math 271 - Quiz #5 **Score:** ______ out of 10.

Name: Key

Show all work clearly and in order. Please box your answers. 10 minutes.

1. Determine whether the given set of functions is linearly independent on the interval $(-\infty, \infty)$. SHOW WORK AND CLEARLY STATE whether the set of functions is **linearly independent** or **linearly dependent**.

   (a) $f_1(x) = \sin(x)$, $f_2(x) = \cos(x)$

$$W(\sin(x), \cos(x)) = \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} = -\sin^2(x) - \cos^2(x) = -(\sin^2(x) + \cos^2(x)) = -1 \neq 0$$

So $\boxed{\text{Linearly Independent}}$

   (b) $g_1(x) = 2$, $g_2(x) = x$, $g_3(x) = 4 + 3x$

SOL 1:

$$2g_1(x) + 3g_2(x) - g_3(x) = 2(2) + 3(x) - (4+3x) = 0$$

NOT all zero $\Rightarrow$ $\boxed{\text{linearly dependent}}$

SOL 2:

$$W(2, x, 4+3x) = \begin{vmatrix} 2 & x & 4+3x \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} - x\begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix} + (4+3x)\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 - 0 + 0 = 0$$

$\boxed{\text{linearly dependent}}$

2. (a) Verify that $y_1 = x^{-1}$ and $y_2 = x^4$ form a fundamental set of solutions of

$$x^2y'' - 2xy' - 4y = 0,$$

on $(0, \infty)$.

(i) Show y_1 is a solution:

$$y_1 = x^{-1}$$
$$y_1' = -x^{-2}$$
$$y_1'' = +2x^{-3}$$
$$x^2y'' - 2xy' - 4y = x^2(2x^{-3}) - 2x(-x^{-2}) - 4(x^{-1})$$
$$= 2x^{-1} + 2x^{-1} - 4x^{-1}$$
$$= 0 \checkmark$$

(ii) Show y_2 is a solution:

$$y_2 = x^4$$
$$y_2' = 4x^3$$
$$y_2'' = 12x^2$$
$$x^2y'' - 2xy' - 4y = x^2(12x^2) - 2x(4x^3) - 4x^4$$
$$= 12x^4 - 8x^4 - 4x^4$$
$$= 0 \checkmark$$

(iii) Show y_1 and y_2 are linearly independent

$$W(y_1, y_2) = W(x^{-1}, x^4) = \begin{vmatrix} x^{-1} & x^4 \\ -x^{-2} & 4x^3 \end{vmatrix} = 4x^{-1}x^3 - (-x^{-2})(x^4)$$
$$= 4x^2 + x^2$$
$$= 5x^2 \neq 0, \text{ if } x \neq 0.$$

(okay on $(0,\infty)$)

(b) Write the general solution of $x^2y'' - 2xy' - 4y = 0$.

Explicit Solution: $\boxed{y = c_1 x^{-1} + c_2 x^4}$