[redacted]

[redacted] Find two power series solutions of

$$y'' + x^2 y = 0$$

about the ordinary point $x=0$.

SOL: Since $x=0$ is an ordinary point we have a solution of the form:

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Thus, $y'' + x^2 y = 0$ becomes

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x^2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{\text{stats } x^0} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+2}}_{\text{stats } x^2} = 0$$

Now take out terms so the series start with the same power of x.

$$2\cdot 1\cdot c_2 x^0 + 3\cdot 2\cdot c_3 x^1 + \sum_{n=4}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

"SHIFT $n=4$ to $n=0$"

$$2c_2 + 6c_3 x + \underbrace{\sum_{n=0}^{\infty} (n+4)(n+3) c_{n+4} x^{n+2}}_{\text{stats at } x^2\ (n=0)} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+2}}_{\text{stats at } x^2\ (n=0)} = 0$$

$$2c_2 + 6c_3 x + \sum_{n=0}^{\infty} \left[(n+4)(n+3) c_{n+4} + c_n \right] x^{n+2} = 0$$

By the identity property

$2c_2 = 0$ AND $6c_3 = 0$ AND $(n+4)(n+3)c_{n+4} + c_n = 0$ for $n = 0, 1, 2, 3, \ldots$

$c_2 = 0$ AND $c_3 = 0$ AND

$$c_{n+4} = \frac{-c_n}{(n+4)(n+3)}$$

always solve for higher coef.

set up the table:

$$c_0 = ?$$
$$c_1 = ?$$
$$c_2 = 0$$
$$c_3 = 0$$

n	$c_{n+4} = \dfrac{-c_n}{(n+4)(n+3)}$
0	$c_4 = \dfrac{-c_0}{4\cdot 3}$
1	$c_5 = \dfrac{-c_1}{5\cdot 4}$
2	$c_6 = \dfrac{-c_2}{6\cdot 5} = 0$
3	$c_7 = \dfrac{-c_3}{7\cdot 6} = 0$
4	$c_8 = \dfrac{-c_4}{8\cdot 7} = \dfrac{c_0}{8\cdot 7\cdot 4\cdot 3}$
5	$c_9 = \dfrac{-c_5}{9\cdot 8} = \dfrac{c_1}{9\cdot 8\cdot 5\cdot 4}$
	$\vdots$

Hence
$$y = \sum_{n=0}^{\infty} c_n x^n$$
$$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$
$$= c_0 + c_1 x + 0x^2 + 0x^3 + \frac{-c_0}{4\cdot 3}x^4 + \frac{-c_1}{5\cdot 4}x^5 + 0x^6 + 0x^7 + \frac{c_0}{8\cdot 7\cdot 4\cdot 3}x^8 + \frac{c_1}{9\cdot 8\cdot 5\cdot 4}x^9$$
$$= \left(c_0 - \frac{c_0}{4\cdot 3}x^4 + \frac{c_0}{8\cdot 7\cdot 4\cdot 3}x^8 + \cdots\right) + \left(c_1 x + \frac{-c_1}{5\cdot 4}x^5 + \frac{c_1}{9\cdot 8\cdot 5\cdot 4}x^9 + \cdots\right)$$
$$= c_0\underbrace{\left(1 - \frac{1}{4\cdot 3}x^4 + \frac{1}{8\cdot 7\cdot 4\cdot 3}x^8 + \cdots\right)}_{y_1} + c_1\underbrace{\left(x - \frac{1}{5\cdot 4}x^5 + \frac{1}{9\cdot 8\cdot 5\cdot 4}x^9 + \cdots\right)}_{y_2}$$

two solutions

$$\boxed{\begin{array}{l} y_1 = 1 - \dfrac{1}{4\cdot 3}x^4 + \dfrac{1}{8\cdot 7\cdot 4\cdot 3}x^8 + \cdots \\ y_2 = x - \dfrac{1}{5\cdot 4}x^5 + \dfrac{1}{9\cdot 8\cdot 5\cdot 4}x^9 + \cdots \end{array}}$$