TEST 2

Math 271 - Differential Equations **Score:** ________ out of 100

Name: ______________________________

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 5 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

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1. The function $y_1 = x^4$ is a solution to $x^2y'' - 7xy' + 16y = 0$. Use the reduction of order equation formula to find a second solution $y_2(x)$. (NOTE: you do not need to vertify that y_1 is a solution, just find y_2.)

$$y'' - \frac{7x}{x^2}y' + \frac{16}{x^2}y = 0 \Rightarrow y'' \underbrace{- \frac{7}{x}}_{P(x)} y' + \frac{16}{x^2}y = 0$$

N.B. $x > 0$ may be assumed.

$$y_2 = x^4 \int \frac{e^{-\int -7/x\,dx}}{(x^4)^2}\,dx$$

$$= x^4 \int \frac{e^{7\ln(x)}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{\ln(x^7)}}{x^8}\,dx$$

$$= x^4 \int \frac{x^7}{x^8}\,dx$$

$$= x^4 \int \frac{1}{x}\,dx = \boxed{x^4 \ln(x)}$$

2. Determine whether the given set of functions is linearly independent on the interval $(0, \infty)$. SHOW WORK AND CLEARLY STATE whether the set of functions is **linearly independent** or **linearly dependent**.

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(a) $f_1(x) = x$, $f_2(x) = x\ln(x)$

$$\begin{vmatrix} x & x\ln(x) \\ 1 & x(\frac{1}{x}) + \ln(x) \end{vmatrix} = x(1 + \ln(x)) - x\ln(x) = x \neq 0 \quad \text{on } (0,\infty)$$

$\boxed{\text{linearly independent}}$

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(b) $g_1(x) = 5$, $g_2(x) = \sin(x)$, $g_3(x) = 10 - 7\sin(x)$

$$2 \cdot g_1 + 7g_2 - g_3 =$$
$$2 \cdot 5 + 7\sin(7) - (10 - 7\sin(x)) = 0$$

$\boxed{\text{lin dependent}}$

3. Complete all of the following parts. **You may not use the auxiliary/characteristic equation method!**

(a) Verify that $y_1 = e^{-x}$ and $y_2 = e^x$ form a fundamental set of solutions of $y'' - y = 0$ on $(-\infty, \infty)$.

Show y_1 is a solution:

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$$\left.\begin{array}{l} y_1 = e^{-x} \\ y_1' = -e^{-x} \\ y_1'' = e^{-x} \end{array}\right\} \Rightarrow y'' - y = e^{-x} - e^{-x} = 0 \checkmark$$

Show y_2 is a solution:

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$$\left.\begin{array}{l} y_2 = e^{x} \\ y_2' = e^{x} \\ y_2'' = e^{x} \end{array}\right\} \Rightarrow y'' - y = e^{x} - e^{x} = 0 \checkmark$$

Show y_1 and y_2 are lin independent:

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$$\begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} = e^{x}e^{-x} - (-e^{-x})(e^{x}) = e^0 + e^0 = 2 \neq 0 \checkmark$$

(b) Verify that $y_p = \frac{1}{8}e^{3x}$ forms a particular solution of $y'' - y = e^{3x}$.

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$$\left.\begin{array}{l} y_p' = \frac{3}{8}e^{3x} \\ y_p'' = \frac{9}{8}e^{3x} \end{array}\right\} \Longrightarrow y'' - y = \frac{9}{8}e^{3x} - \frac{1}{8}e^{3x} = \frac{8}{8}e^{3x} = e^{3x} \checkmark$$

(c) Use (a) and (b) to write the general solution of $y'' - y = e^{3x}$.

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General Solution: $\boxed{y = c_1 e^{-x} + c_2 e^{x} + \frac{1}{8}e^{3x}}$

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4. Find the general solution to the following:

(a) $y'' + y' - 12y = 0$

$m^2 + m - 12 = 0$

$(m+4)(m-3) = 0$

$m = -4 \mid m = 3$

$$\boxed{y = C_1 e^{-4x} + C_2 e^{3x}}$$

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(b) $y''' - 4y'' + 4y' = 0$

$m^3 - 4m^2 + 4m = 0$

$m(m^2 - 4m + 4) = 0$

$m(m-2)(m-2) = 0$

$m = 0, \quad m = 2, \quad m = 2$

$y = C_1 e^{0x} + C_2 e^{2x} + C_3 x e^{2x}$

$$\boxed{y = C_1 + C_2 e^{2x} + C_3 x e^{2x}}$$

10

(c) $y^{(4)} - 16y = 0$

$m^4 - 16 = 0$

$(m^2 - 4)(m^2 + 4) = 0$

$(m-2)(m+2)(m^2+4) = 0$

$m = 2, \quad m = -2 \qquad m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4} = \pm 2i$

$\underbrace{m = 2i, \quad m = -2i}_{}$

$\alpha = 0, \quad \beta = 2$

$y = C_1 e^{2x} + C_2 e^{-2x} + e^{0x}\left[C_3 \sin(2x) + C_4 \cos(2x)\right]$

$$\boxed{y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \sin(2x) + C_4 \cos(2x)}$$

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5. Solve the following differential equation using the <u>method of undetermined coefficients</u>:

$$y'' + y' - 2y = 5e^x$$

<u>Find y_c:</u> for $y'' + y' - 2y = 0$:

$$m^2 + m - 2 = 0$$
$$(m+2)(m-1) = 0$$
$$m = -2, \quad m = 1$$

$$y_c = \underbrace{C_1 e^{-2x}}_{y_1} + \underbrace{C_2 e^{x}}_{y_2}$$

<u>Find y_p:</u>

Looking at $g(x) = 5e^x$:

$$y_p = Ae^x$$

we must adjust since $y_2 = e^x$:

$$y_p = Axe^x$$
$$y_p' = Axe^x + Ae^x$$
$$y_p'' = Axe^x + 2Ae^x$$

$y'' + y' - 2y = 5e^x$ becomes

$$\cancel{Axe^x} + 2Ae^x + \cancel{Axe^x} + Ae^x - \cancel{2Axe^x} = 5e^x$$
$$3Ae^x = 5e^x$$
$$3A = 5$$
$$A = 5/3$$

General Solution: $\boxed{y = C_1 e^{-2x} + C_2 e^x + \tfrac{5}{3} x e^x}$