TEST 1

Math 271 - Differential Equations **Score:** ________ out of 100

5/29/2013 **Name:** Key

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 6 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Fill in the following table with the missing classification information:

ODE	order	linear/nonlinear
$x^3y'' = \cos(y) + (y')^2$	2	nonlinear
$xy''' + \ln(x)y' = 5$	3	linear
$\frac{dA}{dt} - 5A = 0$	1	linear

2. (a) Verify that $y = \ln(x + C)$ is a one-parameter family of solutions to the differential equation $e^y y' = 1$.

$$y' = \frac{1}{x+c}$$

$$e^y y' = e^{\ln(x+c)} \cdot \left(\frac{1}{x+c}\right) = (x+c)\left(\frac{1}{x+c}\right) = 1 \checkmark$$

(b) Use part (a) to find a solution to the initial value problem (IVP) consisting of the differential equation $e^y y' = 1$ and the initial condidtion $y(0) = 0$.

$$y = \ln(x+c)$$

$$y(0) = 0 = \ln(0+c)$$

$$0 = \ln(c)$$

$$C = 1$$

so $\boxed{y = \ln(x+1)}$

3. Find a solution to the following initial-value problem:

$$\frac{dy}{dx} = \frac{y(y+1)}{x^2+1}, \qquad y(0) = 1.$$

Be sure to clearly label steps to maximize your score.

$$\int \frac{dy}{y(y+1)} = \int \frac{dx}{x^2+1}$$

$$\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$$

$$\frac{1}{y(y+1)} = \frac{A(y+1) + By}{y(y+1)}$$

$$1 = A(y+1) + By$$

$$1 = (A+B)y + A$$

$A = 1$ | $A+B=0$, $1+B=0$, $B=-1$

$$\int \left(\frac{1}{y} + \frac{-1}{y+1}\right)dy = \int \frac{dx}{x^2+1}$$

$$\ln|y| - \ln|y+1| = \tan^{-1}(x) + C$$

substituting the initial condition $y'(0) = 1$:

$$\ln|1| - \ln|1+1| = \tan^{-1}(0) + C$$

$$0 - \ln(2) = 0 + C$$

$$C = -\ln(2)$$

Implicit (or Explicit) Solution: $\boxed{\ln|y| - \ln|y+1| = \tan^{-1}(x) - \ln(2)}$

4. (a) Find an <u>explicit</u> solution of:

$$x\frac{dy}{dx} + y = xe^x.$$

Be sure to clearly label steps to maximize your score.

<u>Standard Form</u>: $\frac{dy}{dx} + \frac{1}{x} y = e^x$

<u>Integrating Factor</u>: $e^{\int P(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln|x|} = |x|$

$= x$, if $x > 0$

<u>Multiply</u>:

$$x\left[\frac{dy}{dx} + \frac{1}{x} y\right] = xe^x$$

$$\frac{d}{dx}[x \cdot y] = xe^x$$

<u>Integrate</u>:

$$xy = \int xe^x\,dx$$

$u = x$ | $dv = e^x$

$du = 1\,dx$ | $v = e^x$

$$xy = xe^x - \int e^x\,dx$$

$$xy = xe^x - e^x + C$$

$$y = e^x - \frac{e^x}{x} + \frac{C}{x}$$

Explicit Solution: $\boxed{y = e^x - \frac{e^x}{x} + \frac{C}{x}}$

(b) Give the largest interval over which the general solution is defined. $\boxed{(0, \infty)}$

(c) Are there any transient terms in the general solution? If yes, what are they?

yes, $\frac{C}{x}$

5. Find an explicit solution of:

$$\frac{dy}{dx} - x\sin(x^2)y = 0.$$

Be sure to clearly label steps to maximize your score.

SOL 1: 1^{st} order linear:

Standard Form: ✓

Integrating Factor: $e^{\int P(x)dx} = e^{\int x\sin(x^2)dx}$

$u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\Rightarrow dx = \frac{du}{2x}$

$= e^{\frac{-1}{2}\int \sin(u)\,du}$

$= e^{+1/2\cos(u)}$

$= e^{+1/2\cos(x^2)}$

Multiply:

$$e^{+1/2\cos(x^2)}\left[\frac{dy}{dx} - x\sin(x^2)y\right] = e^{-1/2\cos(x^2)}\cdot 0$$

$$\frac{d}{dx}\left[e^{+1/2\cos(x^2)}\cdot y\right] = 0$$

Integrate:

$$e^{+1/2\cos(x^2)}\cdot y = \int 0\,dx$$

$$e^{+1/2\cos(x^2)}\, y = C$$

$$y = \frac{C}{e^{1/2\cos(x^2)}} = Ce^{-1/2\cos(x^2)}$$

SOL 2: The equation is separable:

$$\frac{dy}{dx} = x\sin(x^2)y$$

$$\int \frac{dy}{y} = \int x\sin(x^2)\,dx$$

$u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow \frac{du}{2x} = dx$

$$\ln|y| = \int \cancel{x}\sin(u)\cdot\frac{du}{2\cancel{x}}$$

$$\ln|y| = \frac{1}{2}\int \sin(u)\,du$$

$$\ln|y| = \frac{1}{2}(-\cos(u)) + C$$

$$\ln|y| = -\frac{1}{2}\cos(x^2) + C$$

$$|y| = e^{-\frac{1}{2}\cos(x^2) + C}$$

$$|y| = De^{-1/2\cos(x^2)}$$

$$y = Ge^{-1/2\cos(x^2)}$$

SAME.

Explicit Solution: $\boxed{y = Ce^{-1/2\cos(x^2)}}$

6. (a) What substitution turns the Bernoulli equation $\frac{dy}{dx} - y = y^2$ into a 1st order linear differential equation?

$n=2$

$u = y^{1-n} = y^{1-2} = y^{-1}$, $\boxed{u = \frac{1}{y}}$ or $\boxed{y = \frac{1}{u}}$

(b) What substitution turns the homogeneous of degree equation $(x^2 - y^2)dx + xydy = 0$ into a separable differential equation?

$\boxed{y = ux}$

(c) Pick one of the two differential equations above to fully solve.

I will solve the differential equation from (a) (b) (CIRCLE ONE)

SOLUTION TO (a)

$y = \frac{1}{u} = u^{-1}$

$\frac{dy}{dx} = -u^{-2}\frac{du}{dx}$

$\frac{dy}{dx} - y = y^2$ becomes

$\left(-u^{-2}\frac{du}{dx}\right) - \left(\frac{1}{u}\right) = \left(\frac{1}{u}\right)^2$

$\frac{du}{dx} - \left(\frac{1}{u}\right)\left(\frac{1}{-u^{-2}}\right) = \left(\frac{1}{u^2}\right)\left(\frac{1}{-u^{-2}}\right)$

$\frac{du}{dx} + u = -1$ ⇐ 1st order linear!

Standard Form: ✓

Integrating Factor: $e^{\int P(x)dx} = e^{\int 1 dx} = e^x$

Multiply: $e^x\left[\frac{du}{dx} + u\right] = -e^x$

Integrate $\frac{d}{dx}[e^x \cdot u] = -e^x$

$e^x \cdot u = -\int e^x$

$e^x \cdot u = -e^x + C$

$u = \frac{-e^x + C}{e^x}$

$u = -1 + \frac{C}{e^x}$

$\boxed{\frac{1}{y} = -1 + \frac{C}{e^x}}$

SOLUTION TO (b)

$y = ux$

$dy = u\,dx + x\,du$

$(x^2 - y^2)dx + xy\,dy = 0$ becomes

$(x^2 - (ux)^2)dx + x(ux)\cancel{dy}(u\,dx + x\,du) = 0$

$(x^2 - u^2x^2)dx + ux^2\cancel{dy}(u\,dx + x\,du) = 0$

$(x^2 - \cancel{u^2x^2} + \cancel{u^2x^2})dx + ux^3du = 0$

$x^2dx = -ux^3du$

$\frac{x^2}{x^3}dx = -u\,du$ ⇐ separable!

$\int \frac{1}{x}dx = \int -u\,du$

$\ln|x| = -\frac{u^2}{2} + C$

$\boxed{\ln|x| = -\frac{(y/x)^2}{2} + C}$

Implicit (or Explicit) Solution: