Math 271 - Fall 2012 - Quiz #4 Score: _______ out of 10.

Name: Key

Show all work clearly and in order. Please box your answers.

1. Solve the following differential equation by using an appropriate substitution:

$$x\frac{dy}{dx} - (1+x)y = xy^2.$$

Standard Form: $\frac{dy}{dx} - \frac{1+x}{x}y = \frac{xy^2}{x} = y^2$ ← This is a Bernoulli Equation with $n=2$

substitute: $u = y^{1-n} = y^{1-2} = y^{-1} = \frac{1}{y}$ so $y = \frac{1}{u}$

Also, $\frac{dy}{dx} = \frac{-1}{u^2}\cdot\frac{du}{dx}$ (by Chain Rule)

Now the DE. becomes: $\left[-\frac{1}{u^2}\cdot\frac{du}{dx}\right] - \frac{1+x}{x}\cdot\left[\frac{1}{u}\right] = \left[\frac{1}{u}\right]^2$ Multiply by $-u^2$ to get:

$\frac{du}{dx} + \underbrace{\left(\frac{1+x}{x}\right)}_{P(x)}u = -1$ ← This is 1st order linear!

Step 1:
Step 2: (Integrating Factor: $e^{\int P(x)dx} = e^{\int(\frac{1+x}{x})dx} = e^{\int(\frac{1}{x}+\frac{x}{x})dx} = e^{\int(\frac{1}{x}+1)dx} = e^{\ln|x|+x} = e^{\ln|x|}\cdot e^x = |x|e^x = xe^x$ (if $x>0$)

Step 3: Multiply 1 and 2: $xe^x\left[\frac{du}{dx} + \left(\frac{1+x}{x}\right)u\right] = xe^x[-1]$

$\frac{d}{dx}[xe^x\cdot u] = -xe^x$

Step 4: Integrate: $xe^x\cdot u = -\int xe^x dx$

$u = x$, $du = 1\cdot dx$ | $dv = e^x$, $v = e^x$

so $xe^x u = -[xe^x - \int e^x dx] = -xe^x + e^x + C$

so $u = \frac{-xe^x + e^x + C}{xe^x}$

$u = -1 + \frac{1}{x} + \frac{C}{xe^x}$

Implicit (or Explicit) Solution: $\boxed{\frac{1}{y} = -1 + \frac{1}{x} + \frac{C}{xe^x}}$

2. (a) Solve the following differential equation by using an appropriate substitution:

$$\frac{dy}{dx} = \frac{x+3y}{3x+y}.$$ ← This is homogeneous (of degree 1)

You can do a substitution of $x = uy$ OR $y = ux$. Both work out fine, so let's try $y = ux$, $dy = u\,dx + x\,du$

$\frac{dy}{dx} = \frac{x+3y}{3x+y} \Rightarrow (3x+y)\,dy = (x+3y)\,dx$

$(3x+ux)(u\,dx + x\,du) = (x+3ux)dx$

$3xu\,dx + 3x^2du + u^2x\,dx + ux^2du = x\,dx + 3ux\,dx$

(Let's bring the dx terms to the left hand side)

$\cancel{3xu\,dx} - x\,dx - \cancel{3ux\,dx} + u^2x\,dx = -3x^2du - ux^2du$

$(-x + u^2x)dx = (-3x^2 - ux^2)du$

$x(-1+u^2)\,dx = x^2(-3-u)\,du$

$\frac{x}{x^2}dx = \frac{-3-u}{u^2-1}du$

so $\int\frac{1}{x}dx = \int\frac{-3-u}{u^2-1}du$

$\frac{-3-u}{u^2-1} = \frac{-3-u}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$

$-3-u = A(u+1) + B(u-1)$

$-3-u = Au + A + Bu - B$

$-3-u = (A+B)u + A - B$

so $A+B = -1$ AND $A - B = -3$

$A = -1-B \longrightarrow (-1-B)-B = -3$

$-2B = -2$

$B = 1$

$A = -2$

$\ln|x| = \int\left(\frac{-2}{u-1} + \frac{1}{u+1}\right)du$

$\ln|x| = -2\ln|u-1| + \ln|u+1| + C$

Implicit (or Explicit) Solution: $\boxed{\ln|x| = -2\ln\left|\frac{y}{x}-1\right| + \ln\left|\frac{y}{x}+1\right| + C}$