Determinants

2×2 CASE:

$$\boxed{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} a_{21}}$$

examples:

(a) $\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 2(-1) - (3)(1) = -2 - 3 = -5$

(b) $W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^x(xe^x + e^x) - xe^x \cdot e^x$

$= xe^{2x} + e^{2x} - xe^{2x}$

$= e^{2x}$

3×3 CASE:

$$\boxed{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$$

N.B. There are many ways to compute determinants, and the above is one of them.

examples:

(a) $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ -1 & 1 & 3 \end{vmatrix} = 1 \begin{vmatrix} 3 & 0 \\ 1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix}$

$= 1(9-0) - 0 + 1(2 - 3(-1))$

$= 9 - 0 + 1 \cdot 5$

$= 14$

(b) $W(1, e^x, xe^x) = \begin{vmatrix} 1 & e^x & xe^x \\ 0 & e^x & xe^x + e^x \\ 0 & e^x & xe^x + 2e^x \end{vmatrix} = 1 \begin{vmatrix} e^x & xe^x + e^x \\ e^x & xe^x + 2e^x \end{vmatrix} - e^x \underbrace{\begin{vmatrix} 0 & xe^x + e^x \\ 0 & xe^x + 2e^x \end{vmatrix}}_{0} + xe^x \underbrace{\begin{vmatrix} 0 & e^x \\ 0 & e^x \end{vmatrix}}_{0}$

$= e^x(xe^x + 2e^x) - e^x(xe^x + e^x)$

$= xe^{2x} + 2e^{2x} - xe^{2x} - e^{2x}$

$= e^{2x}$

4x4 CASE :

$$\boxed{\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}}$$

NxN CASE :

Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}$

The minor M_{ij} of A is the determinant of the matrix with entries from A but deleting row i and column j

The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

The determinant of A (for a fixed i) is :

$$\boxed{\det(A) = |A| = \sum_{j=1}^{N} a_{ij} C_{ij}}$$

This is called the Laplace expansion (or cofactor expansion)

The determinant of A (for a fixed j) is

$$\det(A) = |A| = \sum_{i=1}^{N} a_{ij} C_{ij}$$

NOTE: For the 2x2, 3x3 and 4x4 CASES we used this general formula for a fixed $i=1$.