TEST 4

Math 271 - Differential Equations

Score: ________ out of 100

Name: key

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 2 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. $x = 0$ is an ordinary point of the differential equation:

$$y'' + 2xy' + 2y = 0.$$

Find two linearly independent power series solutions about $x = 0$. You should write down the first <u>three nonzero terms of each series solution</u>.

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_{n=0}^{\infty} n C_n x^{n-1} = \sum_{n=1}^{\infty} n C_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) C_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

Now $\quad y'' + 2xy' + 2y = 0 \quad$ becomes:

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + 2x \sum_{n=1}^{\infty} n C_n x^{n-1} + 2\sum_{n=0}^{\infty} C_n x^n = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}}_{\text{Stats at } x^0} + \underbrace{\sum_{n=1}^{\infty} 2n C_n x^n}_{\text{Stats at } x^1} + \underbrace{\sum_{n=0}^{\infty} 2C_n x^n}_{\text{Stats at } x^0} = 0$$

$$2(2-1)C_2 x^0 + \sum_{n=3}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=1}^{\infty} 2n C_n x^n + 2C_0 x^0 + \sum_{n=1}^{\infty} 2C_n x^n = 0$$

$$2C_2 + 2C_0 + \sum_{n=1}^{\infty} (n+2)(n+1) C_{n+2} x^n + \sum_{n=1}^{\infty} 2n C_n x^n + \sum_{n=1}^{\infty} 2C_n x^n = 0$$

$$2C_2 + 2C_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) C_{n+2} + 2n C_n + 2C_n \right] x^n = 0$$

$2C_2 + 2C_0 = 0$ <u>AND</u> $(n+2)(n+1)C_{n+2} + 2nC_n + 2C_n = 0$ for $n = 1, 2, 3, \dots$

$C_2 = \frac{-2C_0}{2} = -C_0$

$$C_{n+2} = \frac{-2nC_n - 2C_n}{(n+2)(n+1)} = \frac{(-2n-2)C_n}{(n+2)(n+1)} \text{ for } n=1,2,\dots$$

$$= \frac{-2\cancel{(n+1)}C_n}{(n+2)\cancel{(n+1)}} \text{ for } n=1,2,\dots$$

n	$C_{n+2} = \frac{(-2n-2)C_n}{(n+2)(n+1)} = \frac{-2C_n}{(n+2)}$
1	$C_3 = \frac{-2C_1}{(1+2)} = -\frac{2}{3}C_1$
2	$C_4 = \frac{-2C_2}{(2+2)} = \frac{-2}{4}(-C_0) = \frac{C_0}{2}$
3	$C_5 = \frac{-2C_3}{(3+2)} = \frac{-2}{5}C_3 = \frac{-2}{5}\left(-\frac{2}{3}C_1\right) = \frac{4}{15}C_1$

Now we can write the solution:

$y = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \dots$

$= C_0 + C_1x + (-C_0)x^2 + (-\frac{2}{3}C_1)x^3 + (\frac{C_0}{2})x^4 + (\frac{4}{15}C_1)x^5 + \dots$

$= (C_0 - C_0x^2 + \frac{C_0}{2}x^4 + \dots) + (C_1x + -\frac{2}{3}C_1x^3 + \frac{4}{15}x^5 + \dots)$

$= C_0(1 - x^2 + \frac{1}{2}x^4 + \dots) + C_1(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \dots)$

So $\boxed{y_1 = 1 - x^2 + \frac{1}{2}x^4 + \dots}$

$\boxed{y_2 = x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \dots}$

2. $x = 0$ is a regular singular point of the differential equation:

$$3xy'' + y' - y = 0.$$

Find two linearly independent series solutions about $x = 0$. You should write down the first <u>three nonzero terms of each series solution</u>.

$$y = \sum_{n=0}^{\infty} C_n x^{n+r}$$
$$y' = \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-2}$$

Now

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1} - \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) C_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1} - \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

x^{r-1} $\qquad x^{r-1}$ $\qquad x^{r}$

$$3r(r-1)C_0 x^{r-1} + \sum_{n=1}^{\infty} 3(n+r)(n+r-1) C_n x^{n+r-1} + rC_0 x^{r-1} + \sum_{n=1}^{\infty} (n+r) C_n x^{n+r-1} - \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$[3r(r-1)+r] C_0 x^{r-1} + \sum_{n=1}^{\infty} 3(n+r)(n+r-1) \cancel{x} + \sum_{n=1}^{\infty} (n+r) C_n x^{n+r-1} - \sum_{n=1}^{\infty} C_{n-1} x^{n+r-1} = 0$$

$$(3r^2 - 3r + r) C_0 x^{r-1} + \sum_{n=1}^{\infty} \left[3(n+r)(n+r-1) C_n + (n+r) C_n - C_{n-1} \right] x^{n+r-1} = 0$$

$$(3r^2 - 2r) C_0 x^{r-1} + \sum_{n=1}^{\infty} \left[3(n+r)(n+r-1) C_n + (n+r) C_n - C_{n-1} \right] x^{n+r-1} = 0$$

$3r^2 - 2r = 0$
$r(3r - 2) = 0$
$r = 0 \mid r = \frac{2}{3}$

<u>AND</u> $\quad 3(n+r)(n+r-1) C_n + (n+r) C_n - C_{n-1} = 0$ for $n = 1, 2, 3, \ldots$

$$C_n \left[3(n+r)(n+r-1) + (n+r) \right] = C_{n-1}$$

$$C_n = \frac{C_{n-1}}{(n+r)(3(n+r-1)+1)} = \frac{C_{n-1}}{(n+r)(3n+3r-2)} \quad \text{for } n = 1, 2, 3, \ldots$$

$r = 0$

n	$C_n = \frac{C_{n-1}}{(n)(3n-2)}$
1	$C_1 = \frac{C_0}{1(3-2)} = C_0$

$r = 2/3$

n	$C_n = \frac{C_{n-1}}{(n+2/3)(3n)}$
1	$C_1 = \frac{C_0}{(1+2/3)(3)} = \frac{C_0}{5}$

$$y_1 = x^0 \left[C_0 + C_1 x + C_2 x^2 + \cdots \right]$$

$$\boxed{y_1 = x^0 \left[C_0 + C_0 x + \cdots \right]}$$

OR $y_1 = 1 + x + \cdots$

$$y_2 = x^{2/3} \left[C_0 + C_1 x + C_2 x^2 + \cdots \right]$$

$$\boxed{y_2 = x^{2/3} \left[C_0 + \frac{C_0}{5} x + \cdots \right]}$$

OR $y_2 = x^{2/3} \left(1 + \frac{1}{5} x + \cdots \right)$