Math 271 - Differential Equations - Test 3

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2nd Order Linear Equations

Solve:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x). \qquad (0.1)$$

Method:

1. Find y_c (the general solution to the associated homogeneous equation: $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$.)
2. Find y_p (any particular solution of Equation (0.1)).
3. The general solution of Equation (0.1) is

$$y = y_c + y_p.$$

Variation of Parameters (a method for finding y_p):

1. Write Equation (0.1) in standard form to find $f(x)$:

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \underbrace{\frac{g(x)}{a_2(x)}}_{f(x)}.$$

2. Label y_1 and y_2 in your solution for y_c:

$$y_c = C_1 y_1 + C_2 y_2.$$

3. Compute the **Wronskian**:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

4. Solve for u_1 and u_2:

$$u_1 = \int \frac{-y_2 f(x)}{W} dx,$$

$$u_2 = \int \frac{y_1 f(x)}{W} dx.$$

5. Now we can write the particular solution y_p as:

$$y_p = u_1 y_1 + u_2 y_2.$$

2nd Order Linear Equations with Constant Coefficients

Solve:

$$ay'' + by' + cy = g(x), \qquad (0.2)$$

where $a, b,$ and c are constants.

How to find y_c

To find y_c for the homogeneous equation

$$ay'' + by' + cy = 0, \qquad (0.3)$$

find the roots of the **Auxiliary Equation:**

$$am^2 + bm + c = 0. \qquad (0.4)$$

Case I: Two real roots m_1 and m_2.

$$y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}.$$

Case II: One real root m_1 (repeated root).

$$y_c = C_1 e^{m_1 x} + C_2 x e^{m_1 x}.$$

Case III: Complex conjugate roots $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$.

$$y_c = e^{\alpha x}[C_1 \cos(\beta x) + C_2 \sin(\beta x)],$$

or

$$y_c = C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x),$$

How to find y_p

TWO METHODS

If $g(x)$ is polynomial, exponential (e^{kx}), sine ($\sin(kx)$), cosine ($\cos(kx)$), or combinations of these, then we can use:

Method of Undetermined Coefficients:

1. Write the FORM for y_p based on the form of $g(x)$.
2. Adjust the FORM for y_p based on y_c. Specifically, if any part of the form for y_p appears in y_c, adjust by multiplying that portion of the form for y_p by x. Repeat this until no part of y_c appears in your form for y_p and each peice of y_p is linearly independent.
3. Find y_p, y_p' and y_p'' and plug these into Equation (0.2).
4. Solve for the unknown constants in the FORM for y_p to get the final particular solution y_p.

Otherwise, you can always use **Variation of Parameters!**

Cauchy-Euler Equations (2nd Order Linear Equations with Some Special Coefficients)

Solve:

$$ax^2y'' + bxy' + cy = g(x), \qquad (0.5)$$

where $a, b,$ and c are constants.

How to find y_c

To find y_c for the homogeneous equation

$$ax^2y'' + bxy' + cy = 0, \qquad (0.6)$$

find the roots of the **Auxiliary Equation:**

$$am^2 + (b-a)m + c = 0. \qquad (0.7)$$

Case I: Two real roots m_1 and m_2.

$$y_c = C_1 x^{m_1} + C_2 x^{m_2}.$$

Case II: One real root m_1 (repeated root).

$$y_c = C_1 x^{m_1} + C_2 x^{m_1}\ln(x).$$

Case III: Complex conjugate roots $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$.

$$y_c = x^{\alpha}[C_1 \cos(\beta \ln(x)) + C_2 \sin(\beta \ln(x))],$$

or

$$y_c = C_1 x^{\alpha}\cos(\beta \ln(x)) + C_2 x^{\alpha}\sin(\beta \ln(x)),$$

How to find y_p

ONE METHOD

You must use **Variation of Parameters!**

PLAN B: Start All Over and Change to Constant Coefficients

You can always change a Cauchy-Euler Equation into a 2nd order linear equation with constant coefficients via the substitution

$$x = e^t,$$

and then resubstituting $t = \ln(x)$ at the end.

Spring/Mass Systems

Weight:

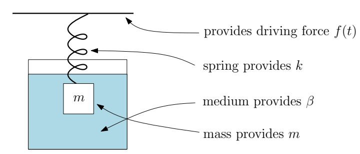
$$w = F_{\text{gravity}} = mg,$$

where m is the **mass** and g is the **acceleration due to gravity**.

Hooke's Law:

$$F_{\text{spring}} = ks,$$

where k is the **spring constant** and s is the **displacement**.



There are 3 main types of spring/mass systems, and all are modeled using a 2nd order linear initial value problem. Note: the following linear differential equations all have constant coefficients.

1. **Free Undamped:**
Find m and k using the above concepts.

$$m\frac{d^2x}{dt^2} + kx = 0,$$
$$x(t_0) = x_0,$$
$$x'(t_0) = x_1.$$

2. **Free Damped:**
Find m and k using the above concepts. β is usually given by the sentence: "...there is a damping force numerically equal to β times the instantaneous velocity...".

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0,$$
$$x(t_0) = x_0,$$
$$x'(t_0) = x_1.$$

3. **Driven Motion with Damping:**
Find m and k using the above concepts. β is usually given by the sentence: "...there is a damping force numerically equal to β times the instantaneous velocity...". The function $f(t)$ is given as the **driving (external) force**.

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t),$$
$$x(t_0) = x_0,$$
$$x'(t_0) = x_1.$$