TEST 2

Math 271 - Differential Equations

Score: ________ out of 100

Name: Key

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 5 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Topics

1. Reduction of Order
2. Linear dependence / independence, Wronskian
3. Fundamental set of solutions, (verify) Wronskian, Particular solution, (verify) General solution: $y = y_c + y_p$
4. Linear, homogeneous with constant coef. Auxiliary / Characteristic Equation Method.
5. Bernoulli Equation Homogeneous (of degree) Equation

1. The function $y_1 = x^2$ is a solution to $y'' + \frac{2}{x}y' - \frac{6}{x^2}y = 0$. Use the reduction of order equation formula to find a second solution $y_2(x)$. (NOTE: you do not need to vertify that y_1 is a solution, just find y_2.)

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2}\,dx$$

+5 $$= x^2 \int \frac{e^{-\int 2/x\,dx}}{(x^2)^2}\,dx$$

+2 $$= x^2 \int \frac{e^{-2\ln|x|}}{x^4}\,dx$$

$$= x^2 \int \frac{e^{\ln|x|^{-2}}}{x^4}\,dx$$

$$= x^2 \int \frac{x^{-2}}{x^4}\,dx$$

+3 $$= x^2 \int x^{-2-4}\,dx = x^2 \int x^{-6}dx = x^2\,\frac{x^{-5}}{-5} = \boxed{\frac{-1}{5}x^{-3}}$$ +3

+2 or just $\boxed{\frac{1}{x^3}}$

2. Determine whether the given set of functions is linearly independent on the interval $(-\infty, \infty)$. SHOW WORK AND CLEARLY STATE whether the set of functions is **linearly independent** or **linearly dependent.**

(a) $f_1(x) = e^x$, $f_2(x) = xe^x$

$$W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^x(xe^x + e^x) - e^x xe^x$$

$$= xe^{2x} + e^{2x} - xe^{2x}$$

$$= e^{2x} \neq 0$$ So $\boxed{\text{linearly independent}}$

(b) $g_1(x) = 2$, $g_2(x) = x^2$, $g_3(x) = 4 - x^2$

SOL 1: Notice that $-2(2) + 1(x^2) + 1(4 - x^2) = 0$

that is $-2\,g_1(x) + 1\,g_2(x) + 1\,g_3(x) = 0$

Not all 0, so $\boxed{\text{linearly dependent}}$

SOL 2: Use the Wronskian:

$$W(2, x^2, 4-x^2) = \begin{vmatrix} 2 & x^2 & 4-x^2 \\ 0 & 2x & -2x \\ 0 & 2 & -2 \end{vmatrix} = 2\begin{vmatrix} 2x & -2x \\ 2 & -2 \end{vmatrix} - x^2\begin{vmatrix} 0 & -2x \\ 0 & -2 \end{vmatrix} + (4-x^2)\begin{vmatrix} 0 & 2x \\ 0 & 2 \end{vmatrix}$$

$$= 2(-4x + 4x) - 0 + 0 = 0$$ so $\boxed{\text{linearly dependent}}$

3. Complete all of the following parts. **You may not use the auxiliary/characteristic equation method!**

(a) Verify that $y_1 = e^{-x}$ and $y_2 = e^{2x}$ form a fundamental set of solutions of $y'' - y' - 2y = 0$ on $(-\infty, \infty)$.

(i) e^{-x} is a solution:

$$\left.\begin{aligned} y &= e^{-x} \\ y' &= -e^{-x} \\ y'' &= e^{x} \end{aligned}\right\} \text{ so } \quad \begin{aligned} y'' - y' - 2y &= e^{x} - (-e^{x}) - 2e^{-x} \\ &= e^{x} + e^{x} - 2e^{-x} \\ &= 0 \checkmark \end{aligned}$$

(ii) e^{2x} is a solution:

$$\left.\begin{aligned} y &= e^{2x} \\ y' &= 2e^{2x} \\ y'' &= 4e^{2x} \end{aligned}\right\} \text{ so } \quad y'' - y' - 2y = 4e^{2x} - 2e^{2x} - 2e^{2x} = 0 \checkmark$$

(iii) e^{-x} and e^{2x} are linearly independent on $(-\infty, \infty)$

$$W(e^{-x}, e^{2x}) = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = e^{-x}(2e^{2x}) - (-e^{-x})(e^{2x})$$
$$= 2e^{x} + e^{x}$$
$$= 3e^{x} \neq 0$$

(b) Verify that $y_p = \sin(2x)$ forms a particaular solution of $y'' - y' - 2y = -6\sin(2x) - 2\cos(2x)$.

$$\left.\begin{aligned} y_p &= \sin(2x) \\ y_p' &= 2\cos(2x) \\ y_p'' &= -4\sin(2x) \end{aligned}\right\} \text{ So } \quad \begin{aligned} y_p'' - y_p' - 2y_p &= -4\sin(2x) - 2\cos(2x) - 2\sin(2x) \\ &= -6\sin(2x) - 2\cos(2x) \checkmark \end{aligned}$$

(c) Use (a) and (b) to write the general solution of $y'' - y' - 2y = -6\sin(2x) - 2\cos(2x)$.

General Solution: $\boxed{y = c_1 e^{-x} + c_2 e^{2x} + \sin(2x)}$

4. Find the general solution to the following:

(a) $y'' - 6y' - 16y = 0$

$m^2 - 6m - 16 = 0$

$(m-8)(m+2) = 0$

$m = 8 \quad | \quad m = -2$

so the general solution is:

$$\boxed{y = c_1 e^{8x} + c_2 e^{-2x}}$$

(b) $y''' + 8y'' + 15y' = 0$

$m^3 + 8m^2 + 15m = 0$

$m(m^2 + 8m + 15) = 0$

$m(m+5)(m+3) = 0$

$m = 0 \quad | \quad m = -5 \quad | \quad m = -3$

so the general solution is

$y = c_1 e^{0x} + c_2 e^{-5x} + c_3 e^{-3x}$

$$\boxed{y = c_1 + c_2 e^{-5x} + c_3 e^{-3x}}$$

(c) $y^{(4)} - y = 0$

$m^4 - 1 = 0$

$(m^2 - 1)(m^2 + 1) = 0$

$(m-1)(m+1)(m^2+1) = 0$

$m = 1 \quad | \quad m = -1 \quad | \quad m^2 + 1 = 0$

$m^2 = -1$

$m = \pm\sqrt{-1} = \pm i$

$m = 0 \pm i$

so $\alpha = 0$ $\quad \beta = 1$

so the general solution is

$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{0 \cdot x} \sin(1 \cdot x) + c_4 e^{0 \cdot x} \cos(1 \cdot x)$

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} + c_3 \sin(x) + c_4 \cos(x)}$$

5. (a) What substitution turns $\frac{dy}{dx} - y = e^x y^2$ into a 1st order linear differential equation?

$$u = y^{1-2} \Rightarrow \boxed{u = y^{-1} = \frac{1}{y}} \quad \text{OR} \quad \boxed{y = \frac{1}{u}}$$

(b) What substitution turns $ydx = 2(x+y)dy$ into a separable differential equation?

$$\boxed{u = \frac{x}{y}} \quad \text{OR} \quad \boxed{u = \frac{y}{x}}$$

(c) Using the substitution you indicated in either (a) or (b) find the general solution of the corresponding differential equation.

I will solve (a) (b) (CIRCLE ONE)

solution for (a): substitute $y = \frac{1}{u} \Rightarrow \frac{dy}{dx} = \frac{-1}{u^2}\frac{du}{dx}$

Thus, $\frac{dy}{dx} - y = e^x y^2$ becomes $\left[-\frac{1}{u^2}\cdot\frac{du}{dx}\right] - \left[\frac{1}{u}\right] = e^x\left[\frac{1}{u}\right]^2$

$$\frac{-1}{u^2}\frac{du}{dx} - \frac{1}{u} = \frac{e^x}{u^2}$$

$$\frac{du}{dx} + \frac{u^2}{u} = \frac{-e^x u^2}{u^2}$$

$$\frac{du}{dx} + u = -e^x \quad (1^{st} \text{ order linear})$$

Step 1: Standard Form DONE!

Step 2: Integrating Factor: $e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$

Step 3: Multiply steps 1 and 2:

$$e^x\left[\frac{du}{dx} + u\right] = e^x\left[-e^x\right]$$

$$\frac{d}{dx}\left[e^x \cdot u\right] = -e^{2x}$$

Step 4: Integrate

$$e^x \cdot u = \int -e^{2x}\,dx$$

$$e^x \cdot u = -\frac{e^{2x}}{2} + C$$

$$u = -\frac{e^{2x}}{2e^x} + \frac{C}{e^x}$$

$$u = -\frac{e^x}{2} + \frac{C}{e^x}$$

$$\frac{1}{y} = \frac{-e^x}{2} + \frac{C}{e^x}$$

Implicit (or Explicit) Solution: $\boxed{\frac{1}{y} = \frac{-e^x}{2} + \frac{C}{e^x}}$

Solution for (b) :

$$y\,dx = 2(x+y)\,dy$$

Sol 1:

$$x = uy$$
$$dx = u\,dy + y\,du$$

$$y(u\,dy + y\,du) = 2(uy+y)\,dy$$
$$yu\,dy + y^2\,du = 2y(u+1)\,dy$$
$$y^2\,du = 2y(u+1)\,dy - yu\,dy$$
$$y^2\,du = [2y(u+1) - yu]\,dy$$
$$y^2\,du = [y(2(u+1) - u)]\,dy$$
$$y^2\,du = [y(2u+2-u)]\,dy$$
$$y^2\,du = y(u+2)\,dy$$
$$y\,du = (u+2)\,dy$$
$$\frac{du}{u+2} = \frac{dy}{y}$$
$$\int \frac{du}{u+2} = \int \frac{dy}{y}$$
$$\ln|u+2| = \ln|y| + C$$
$$\boxed{\ln\left|\frac{x}{y}+2\right| = \ln|y| + C}$$

Sol 2:

$$y = ux$$
$$dy = u\,dx + x\,du$$

$$ux\,dx = 2(x+ux)(u\,dx + x\,du)$$
$$ux\,dx = 2(xu\,dx + x^2\,du + u^2x\,dx + ux^2\,du)$$
$$ux\,dx = 2xu\,dx + 2x^2\,du + 2u^2x\,dx + 2ux^2\,du$$
$$ux\,dx - 2xu\,dx - 2u^2x\,dx = 2x^2\,du + 2ux^2\,du$$
$$(-xu - 2u^2x)\,dx = (2x^2 + 2ux^2)\,du$$
$$x(-u-2u^2)\,dx = 2x^2(1+u)\,du$$
$$-xu(1+2u)\,dx = 2x^2(1+u)\,du$$

[redacted]

$$-\frac{1}{x}\,dx = \frac{2(u+1)}{u(1+2u)}\,du$$
$$\int -\frac{1}{x}\,dx = 2\int \frac{(1+u)}{u(1+2u)}\,du$$
$$\int -\frac{1}{x}\,dx = 2\int\left[\frac{1}{u} - \frac{1}{2u+1}\right]du$$
$$-\ln|x| + C = 2\left[\ln|u| - \frac{1}{2}\ln|2u+1|\right]$$
$$\boxed{-\ln|x| + C = 2\ln\left|\frac{y}{x}\right| - \frac{1}{2}\ln\left|\frac{2y}{x}+1\right|}$$