TEST 1

Math 271 - Differential Equations

Score: ________ out of 100

9/20/2012

Name: Key

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 5 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

What was on this test?

1) Classification of ODEs
2) Verify a solution to an ODE
   Find a solution to an IVP using the verified solution
3) Separable ODE / Integration By Partial Fractions
4) 1st ORDER LINEAR ODE / Integration By Parts
5) Separable ODE – OR – 1st ORDER LINEAR / u-substitution

1. Fill in the following table with the missing classification information:

ODE	order	linear/nonlinear
$\sin(x)y'' = \sqrt{\ln(y) + (y')^2}$	2	nonlinear
$\cos(\theta)y''' + \sin(\theta)y' = 3\ln(\theta)$	3	linear
$\frac{dP}{dt} = 5P$	1	linear

2. (a) Verify that $P = \frac{Ce^t}{1+Ce^t}$ is a one-parameter family of solutions to the differential equation $\frac{dP}{dt} = P(1-P)$.

$$\text{LHS} = \frac{dP}{dt} = \frac{(1+Ce^t)\cdot Ce^t - Ce^t \cdot Ce^t}{(1+Ce^t)^2} = \frac{Ce^t + \cancel{(Ce^t)^2} - \cancel{(Ce^t)^2}}{(1+Ce^t)^2}$$

$$= \frac{Ce^t}{(1+Ce^t)^2}$$

SAME !

$$\text{RHS} = P(1-P) = \frac{Ce^t}{1+Ce^t}\left[1 - \frac{Ce^t}{1+Ce^t}\right]$$

$$= \frac{Ce^t}{(1+Ce^t)}\left(\frac{1+\cancel{Ce^t}-\cancel{Ce^t}}{(1+Ce^t)}\right) = \frac{Ce^t}{(1+Ce^t)^2}$$

(b) Use part (a) to find a solution to the initial value problem (IVP) consisting of the differential equation $\frac{dP}{dt} = P(1-P)$ and the initial condidtion $P(0) = 4$.

$$P = \frac{Ce^t}{(1+Ce^t)}$$

$$P(0) = \frac{Ce^0}{(1+Ce^0)} = \frac{C\cdot 1}{(1+C\cdot 1)} = \frac{C}{(1+C)} = 4$$

$$C = 4(1+C)$$

$$C = 4+4C$$

$$-3C = 4$$

$$C = 4/{-3}$$

$$\boxed{P = \frac{(\frac{4}{-3})e^t}{1+(\frac{4}{-3})e^t}}$$

3. (a) Classify the following differential equation: $\frac{dy}{dx} = y(y-2)e^x$

i. ORDER: $\boxed{1}$

ii. LINEAR/NONLINEAR: $\boxed{\text{NONLINEAR}}$

iii. SEPARABLE/NOT SEPARABLE: $\boxed{\text{SEPARABLE}}$

(b) Use your classification from (a) to use the appropriate method in the following problem. Be sure to clearly label steps to maximize your score.

Find a solution to the following initial-value problem:

$$\frac{dy}{dx} = y(y-2)e^x, \qquad y(0) = 1.$$

$$\int \frac{dy}{y(y-2)} = \int e^x \, dx$$

NEED Partial Fractions to integrate:

$$\frac{1}{y(y-2)} = \frac{A}{y} + \frac{B}{y-2} = \frac{A(y-2) + By}{y(y-2)} = \frac{Ay - 2A + By}{y(y-2)}$$

$$1 = (A+B)y \quad -2A$$

$A + B = 0$ | $-2A = 1$

$B = -A$ | $A = -1/2$

$B = 1/2$

$$\frac{1}{y(y-2)} = \frac{-1/2}{y} + \frac{1/2}{y-2} \quad \underline{\text{SO}}$$

$$\int \left[\frac{-1/2}{y} + \frac{1/2}{y-2}\right] = \int e^x$$

$$-\tfrac{1}{2}\ln|y| + \tfrac{1}{2}\ln|y-2| = e^x + C$$

plug in $y=1$, $x=0$ $\Rightarrow$ $\underbrace{-\tfrac{1}{2}\ln|1|}_{0} + \underbrace{\tfrac{1}{2}\ln|1-2|}_{0} = e^0 + C$

$0 + 0 = 1 + C \Rightarrow C = -1$

Implicit (or Explicit) Solution: $\boxed{-\frac{1}{2}\ln|y| + \frac{1}{2}\ln|y-2| = e^x - 1}$

4. (a) Classify the following differential equation: $x\frac{dy}{dx} + y = x\sin(x)$.

i. ORDER: $\boxed{1}$

ii. LINEAR/NONLINEAR: LINEAR

iii. SEPARABLE/NOT SEPARABLE: NOT SEPARABLE

(b) Use your classification from (a) to use the appropriate method in the following problem. Be sure to clearly label steps to maximize your score.

Find an explicit solution of:

$$x\frac{dy}{dx} + y = x\sin(x).$$

1) Standard Form:

$$\frac{dy}{dx} + \frac{1}{x}\cdot y = \frac{\cancel{x}\sin(x)}{\cancel{x}}$$

$$\frac{dy}{dx} + \underbrace{\frac{1}{x}}_{P(x)}\cdot y = \sin(x)$$

2) Integrating Factor (I.F.):

$$e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x| = x$$

assume $x>0$, so

3) Multiply (1) and (2)

$$x\left[\frac{dy}{dx} + \frac{1}{x}\cdot y\right] = x\sin(x)$$

$$\frac{d}{dx}[x\cdot y] = x\sin(x)$$

4) Integrate Both Sides:

$$x\cdot y = \int x\sin(x)\,dx$$

NEED integration by parts

$u = x$, $du = 1dx$ || $dv = \sin(x)$, $v = -\cos(x)$

$$x\cdot y = -x\cos(x) - \int(-\cos(x))\cdot dx$$

$$x\cdot y = -x\cos(x) + \sin(x) + C$$

Explicit Solution: $\boxed{y = -\cos(x) + \frac{\sin(x)}{x} + \frac{C}{x}}$

(c) Give the largest interval over which the general solution is defined. $\boxed{(0,\infty)}$

(d) Are there any transient terms in the general solution? If yes, what are they?

YES, $\frac{\sin(x)}{x}$ AND $\frac{C}{x}$

5. (a) Classify the following differential equation: $\frac{dy}{dx} - y\cos(x)e^{\sin(x)} = 0.$

i. ORDER: $\boxed{1}$

ii. LINEAR/NONLINEAR: $\boxed{\text{LINEAR}}$

iii. SEPARABLE/NOT SEPARABLE: $\boxed{\text{SEPARABLE}}$

(b) Use your classification from (a) to use the appropriate method in the following problem. Be sure to clearly label steps to maximize your score.

Find an explicit solution of:

$$\frac{dy}{dx} - y\cos(x)e^{\sin(x)} = 0.$$

SOLUTION 1: (separable method)

$$\frac{dy}{dx} = y\cos(x)e^{\sin(x)}$$

$$\int \frac{dy}{y} = \int \cos(x)e^{\sin(x)}$$

u-substitution:

$u = \sin(x)$

$\frac{du}{dx} = \cos(x)$

$dx = \frac{du}{\cos(x)}$

$$\ln|y| = \int \cancel{\cos(x)} \cdot e^u \cdot \frac{du}{\cancel{\cos(x)}}$$

$$\ln|y| = \int e^u du$$

$$\ln|y| = e^u + C$$

$$\ln|y| = e^{\sin(x)} + C$$

$$|y| = e^{(e^{\sin(x)} + C)}$$

$$|y| = e^{e^{\sin(x)}} \cdot \underbrace{e^C}_{A}$$

$$|y| = Ae^{e^{\sin(x)}}$$

$$y = Be^{e^{\sin(x)}}$$

SOLUTION 2: (1st order linear method)

1) Standard Form: DONE!

2) Integrating Factor (I.F.):

$$e^{\int P(x)dx} = e^{\int(-\cos(x)e^{\sin(x)})dx}$$

u-substitution

$u = \sin(x)$

$\frac{du}{dx} = \cos(x) \Rightarrow dx = \frac{du}{\cos(x)}$

$$= e^{\int -\cancel{\cos(x)}e^u \cdot \frac{du}{\cancel{\cos(x)}}}$$

$$= e^{\int -e^u du}$$

$$= e^{-e^u} = e^{-e^{\sin(x)}}$$

3) Multiply (1) and (2):

$$e^{-e^{\sin(x)}}\left[\frac{dy}{dx} - y\cos(x)e^{\sin(x)}\right] = e^{-e^{\sin(x)}} \cdot 0 = 0$$

$$\frac{d}{dx}\left[e^{-e^{\sin(x)}} \cdot y\right] = 0$$

4) Integrate:

$$e^{-e^{\sin(x)}} \cdot y = C$$

$$y = \frac{C}{e^{-e^{\sin(x)}}} = Ce^{e^{\sin(x)}}$$

Explicit Solution: $\boxed{y = Be^{e^{\sin(x)}}}$