# INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

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## 1. Partial Fractions

A **rational function** is a ratio of polynomials. In other words, a function of the form:

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}, \text{ where } q(x) \neq 0.$$

In this section we will show how to integrate any rational function. This is done by expressing the rational function as a sum of simpler fractions, called **partial fractions**, which are simple to integrate.

To illistate this idea, consider the following:

$$\frac{1}{x+1} - \frac{3}{x-1} = \frac{1(x-1) - 3(x+1)}{(x-1)(x+1)} = \frac{-2x-4}{x^2-1}.$$

Reversing this procedure leads to a nice solution to the integral of the rational function on the right:

$$\begin{aligned}\int \frac{-2x-4}{x^2-1}dx &= \int \left(\frac{1}{x+1} - \frac{3}{x-1}\right) dx \\ &= \ln|x+1| - 3\ln|x-1| + C.\end{aligned}$$

So let's begin the journey of integrating an arbitrary rational function. As you might notice, for a general rational function there are many cases to consider.

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**Step 1:** If $\deg(p(x)) \geq \deg(q(x))$ we must perform polynomial long division until a remainder $r(x)$ is obtained where $\deg(r(x)) < \deg(q(x))$. We would then have:

$$f(x) = \frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}.$$

Note that both $s(x)$ and $r(x)$ are polynomials.

**Example 1.1.** Evaluate:

$$\int \frac{x^3-5}{x+1}dx.$$

Notice that the degree of the numerator is 3 and the degree of the denominator is 1 so we must perform the "preliminary step" described above.

$$
\begin{array}{rl}
 & x^2 - x + 1 \\
x+1 \big) & \overline{x^3 \qquad\qquad - 5} \\
 & \underline{- x^3 - x^2} \\
 & - x^2 \\
 & \underline{x^2 + x} \\
 & x - 5 \\
 & \underline{- x - 1} \\
 & - 6
\end{array}
$$

Therefore,

$$
\begin{aligned}
\int \frac{x^3 - 5}{x + 1} dx &= \int \left( x^2 - x + 1 - \frac{6}{x+1} \right) dx \\
&= \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - 6 \ln|x + 1| + C.
\end{aligned}
$$

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**Step 2:** Factor $q(x)$ as much as possible.

**Lemma 1.2.** Any polynomial $q(x)$ can be factored as a product of **linear factors** (of the form $ax + b$) and irreducible **quadratic factors** (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

**Example 1.3.**

$$q(x) = x^4 - 81 = (x^2 - 9)(x^2 + 9) = (x - 3)(x + 3)(x^2 + 9).$$

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**Step 3:** Express the rational function $\frac{r(x)}{q(x)}$ (from Step 1) as a sum of **partial fractions** of the form

$$\frac{A}{(ax + b)^j} \text{ or } \frac{Ax + B}{(ax^2 + bx + c)^k}$$

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Now we describe the several cases that may occur in this process.

**CASE 1:** *The denominator, $q(x)$, is a product of distinct linear factors* (This is where you are very happy).

So $q(x)$ can be written in the following form:

$$q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_ix + b_i).$$

Were no factor is repeated (or a constant multiple of another). In this we know there exists constants $A_1, A_2, \ldots, A_i$ such that

$$\frac{r(x)}{q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots \frac{A_i}{a_ix + b_i}$$

**Example 1.4.** Evaluate:

$$\int \frac{x - 3}{x^2 + 5x + 6} dx.$$

*Solution:*
*Step 1:* Done, degree of numerator is 1, degree of denominator is 2.
*Step 2:* $x^2 + 5x + 6 = (x + 3)(x + 2)$.

*Step 3:* Notice the factors $(x+3)$ are distinct $(x+2)$. Therefore, this is an integral of the type from CASE 1.

We want to write:

$$\frac{x-3}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2}.$$

So we need to find $A$ and $B$. With some algebra we have:

$$\frac{x-3}{(x+3)(x+2)} = \frac{A(x+2)}{(x+3)(x+2)} + \frac{B(x+3)}{(x+3)(x+2)} = \frac{A(x+2)+B(x+3)}{(x+3)(x+2)}.$$

So we need to solve

$$x-3 = A(x+2) + B(x+3).$$

Expand and compare coefficients:

$$x-3 = Ax + 2A + Bx + 3B = (A+B)x + (2A+3B).$$

The coefficients must match and therefore:

$$\begin{aligned} A+B &= 1 \\ 2A+3B &= -3. \end{aligned}$$

Solving this we can write $A = 1-B$. Substituting into the second equation we have $-3 = 2(1-B)+3B = 2-2B+3B = 2+B$. Hence, $B=-5$. So, $A = 1-(-5) = 6$. Finally,

$$\frac{x-3}{(x+3)(x+2)} = \frac{6}{x+3} + \frac{-5}{x+2}$$

So we can now write:

$$\begin{aligned} \int \frac{x-3}{x^2+5x+6}dx &= \int \left(\frac{6}{x+3} + \frac{-5}{x+2}\right) dx \\ &= 6\ln|x+3| - 5\ln|x+2| + C. \end{aligned}$$

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**CASE 2:** *$q(x)$ is a product of linear factors, some of which are repeated.*

Suppose that the first linear factor $(a_1x+b_1)$ is repeated say $j$ times. That is, $(a_1x+b_1)^j$ occurs in the factorization of $q(x)$.

If this happens, instead of the single term $\frac{A_1}{a_1x+b_1}$ as in CASE 1 we use:

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_j}{(a_1x+b_1)^j}.$$

For instance, we could write:

$$\frac{2x^2+x-2}{x^3(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{(x+1)^2}.$$

**Example 1.5.** Evaluate:

$$\int \frac{2x^2+x-2}{x^3+x^2}dx.$$

*Step 1:* Done, degree of numerator is 2, degree of denominator is 3.
*Step 2:* $x^3+x^2 = x^2(x+1)$.

*Step 3:* We have a repeated linear factor and a distinct linear factor in the denominator. This is an example of CASE 2 and CASE 1 combined. So we write:

$$\frac{2x^2+x-2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}.$$

Finding a common denominator we have:

$$\begin{aligned}\frac{2x^2+x-2}{x^2(x+1)} &= \frac{Ax(x+1)}{x^2(x+1)} + \frac{B(x+1)}{x^2(x+1)} + \frac{Cx^2}{x^2(x+1)}\\ &= \frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}\\ &= \frac{Ax^2+Ax+Bx+B+Cx^2}{x^2(x+1)}\\ &= \frac{(A+C)x^2+(A+B)x+B}{x^2(x+1)}\end{aligned}$$

So comparing numerators we must have:

$$2x^2+x-2 = (A+C)x^2+(A+B)x+B.$$

Therefore,

$$\begin{aligned}A+C&=2\\ A+B&=1\\ B&=-2\end{aligned}$$

This says $A = 1-(-2) = 3$ and $C = 2-A = 2-3 = -1$.

Finally,

$$\begin{aligned}\int \frac{2x^2+x-2}{x^3+x^2}dx = \int \left(\frac{3}{x} - 2x^{-2} + \frac{-1}{x+1}\right)dx \quad &= 3\ln|x| - 2\frac{x^{-1}}{-1} - \ln|x+1| + C\\ &= 3\ln|x| + \frac{2}{x} - \ln|x+1| + C\end{aligned}$$

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**CASE 3:** *$q(x)$ contains irreducible quadratic factors, none of which is repeated.* Here we must introduce partial fractions of the form

$$\frac{Ax+B}{ax^2+bx+c}.$$

For instance, we can write

$$\frac{x^3-x+1}{(x-2)(x^2+1)} = \frac{A}{(x-2)} + \frac{Bx+C}{x^2+1}.$$

**Example 1.6.** Evaluate:

$$\int \frac{x^3-x+1}{x^3-2x^2+x-2}dx.$$

*Step 1:*

$$\begin{array}{r|l} & 1 \\ \hline x^3 - 2x^2 + x - 2 & x^3 \qquad\quad - x + 1 \\ & -x^3 + 2x^2 \ \ - x + 2 \\ \hline & 2x^2 - 2x + 3 \end{array}$$

So

$$\frac{x^3 - x + 1}{x^3 - 2x^2 + x - 2} = 1 + \frac{2x^2 - 2x + 3}{x^3 - 2x^2 + x - 2}.$$

*Step 2:* $x^3 - 2x^2 + x - 2 = x^2(x-2) + (x-2) = (x-2)(x^2+1)$.
*Step 3:*

$$\frac{2x^2 - 2x + 3}{(x-2)(x^2+1)} = \frac{A}{(x-2)} + \frac{Bx + C}{x^2+1}.$$

So

$$\begin{aligned} \frac{2x^2 - 2x + 3}{(x-2)(x^2+1)} &= \frac{A(x^2+1) + (Bx+C)(x-2)}{(x-2)(x^2+1)} \\ &= \frac{Ax^2 + A + Bx^2 - 2Bx + Cx - 2C}{(x-2)(x^2+1)} \\ &= \frac{(A+B)x^2 + (-2B+C)x - 2C + A}{(x-2)(x^2+1)} \end{aligned}$$

Therefore,

$$\begin{aligned} A + B &= 2 \\ -2B + C &= -2 \\ -2C + A &= 3 \end{aligned}$$

Solving this we have $A = 7/5$, $B = 3/5$ and $C = -4/5$. Finally,

$$\begin{aligned} \int \frac{x^3 - x + 1}{x^3 - 2x^2 + x - 2} dx &= \int \left( 1 + \frac{\frac{7}{5}}{(x-2)} + \frac{\frac{3}{5}x + \frac{-4}{5}}{x^2+1} \right) dx \\ &= x + \frac{7}{5} \ln|x-2| + \frac{3}{5} \int \frac{x}{x^2+1} dx - \frac{4}{5} \int \frac{1}{x^2+1} dx \\ &= x + \frac{7}{5} \ln|x-2| + \frac{3}{5} \frac{\ln|x^2+1|}{2} - \frac{4}{5} \tan^{-1}(x) + C \\ &= x + \frac{7}{5} \ln|x-2| + \frac{3}{10} \ln(x^2+1) - \frac{4}{5} \tan^{-1}(x) + C \end{aligned}$$

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**CASE 4:** *$q(x)$ contains repeated irreducible quadratic factors.* Say $q(x)$ has a factor $ax^2 + bx + c)^i$, where $b^2 - 4ac < 0$, then instead of a single partial fraction as in CASE 3, we must have

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots \frac{A_i x + B_i}{(ax^2 + bx + c)^i}$$

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