8.1 INTEGRATION BY PARTS

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1. Integration by Parts

Recall:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Integrate both sides

$$f(x)g(x) + C = \int [f'(x)g(x) + f(x)g'(x)]dx$$
$$f(x)g(x) + C = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

so

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)dx + C.$$

Let's ignore the arbitrary constant since the remaining integral will introduce an abritrary constant anyways.

If we let $u = f(x)$ and $v = g(x)$ then:

$$\boxed{\int udv = uv - \int vdu.}$$

For definite integrals:

$$\boxed{\int_a^b udv = [uv]_a^b - \int_a^b vdu.}$$

Example 1.1. $\int x\cos(x)dx$. $u = x$, $dv = \cos(x)dx$
$du = dx$, $v = \sin(x)$
and so

$$\begin{aligned}\int x\cos(x)dx &= x\sin(x) - \int \sin(x)dx \\ &= x\sin(x) - (-\cos(x)) + C \\ &= x\sin(x) + \cos(x) + C.\end{aligned}$$

Sometimes we must do the method twice in the integration.

Example 1.2. $\int x^2e^xdx$

$$\begin{array}{r|l} u = x^2 & dv = e^xdx \\ du = 2xdx & v = e^x \end{array}$$

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and so

$$\begin{aligned}\int x^2e^x dx &= x^2e^x - \int e^x 2x dx\\ &= x^2e^x - 2\int xe^x dx.\end{aligned}$$

We must use integratin by parts again.
here

$$\begin{array}{r|l} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array}$$

so

$$\begin{aligned}\int x^2e^x dx &= x^2e^x - 2\int xe^x dx\\ &= x^2e^x - 2\left[xe^x - \int e^x dx\right]\\ &= x^2e^x - 2\left[xe^x - e^x + C\right]\\ &= x^2e^x - 2xe^x + 2e^x + D.\end{aligned}$$

Sometimes we use integration by parts and $dv = du$ and $u =$the rest...

Example 1.3. *Consider* $\int \ln(x)dx$ *or* $\int \tan^{-1}(x)dx$.

Sometimes we use integration by parts and the integral we are solving for appears...then we use algebra!

Example 1.4. $\int e^x \sin(x)dx$. *Let*

$$\begin{array}{r|l} u = e^x & dv = \sin(x)dx \\ du = e^x dx & v = -\cos(x) \end{array}$$

$$\begin{aligned}\int e^x \sin(x)dx &= e^x(-\cos(x)) - \int (-\cos(x))e^x dx\\ &= -e^x\cos(x) + \int e^x\cos(x)dx\end{aligned}$$

Use integration by parts again:

$$\begin{array}{r|l} u = e^x & dv = \cos(x)dx \\ du = e^x dx & v = \sin(x) \end{array}$$

$$\begin{aligned}\int e^x \sin(x)dx &= -e^x\cos(x) + \int e^x\cos(x)dx\\ &= -e^x\cos(x) + \left[e^x\sin(x) - \int \sin(x)e^x dx\right]\\ &= -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)dx.\end{aligned}$$

Notice we have the integral we are trying to solve on the right. Bring it over to the left, join the party via algebra. So we have:

$$2\int e^x \sin(x)dx = -e^x\cos(x) + e^x\sin(x).$$

Therefore,

$$\int e^x \sin(x)dx = \frac{-e^x \cos(x) + e^x \sin(x)}{2} + C.$$

Example 1.5. $\int_1^e \ln(x)dx$.

Let

$$\begin{array}{r|l} u = \ln(x) & dv = dx \\ du = \frac{1}{x}dx & v = x \end{array}$$

$$\begin{aligned} \int_1^e \ln(x)dx &= [x\ln(x)]_1^e - \int_1^e x \cdot \frac{1}{x}dx \\ &= [x\ln(x)]_1^e - \int_1^e 1dx \\ &= [x\ln(x) - x]_1^e \\ &= (e\ln(e) - e) - (1\ln(1) - 1) \\ &= (e - e) - (0 - 1) = 1. \end{aligned}$$

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