Newton

Newton is a program that explores the dynamics of applying Newton's method for finding zeros of complex valued functions. There are two modes:
1. **Time to Converge**, which graphically displays the results of applying Newton's method to a function for various initial seeds. Points in the complex plane are colored according to how long it takes for the distance between two successive Newton iterates to be within a specified tolerance level.
2. **Root Convergence**, which also displays the results of applying Newton's method to a function for various initial seeds. However, in this mode points are colored according to which root Newton's method appears to be converging.
3. **Blend**, which also displays the results of applying Newton's method to a function for various initial seeds. Points are colored according to which root Newton's method appears to be converging, but the colors are shaded darker the more iterations needed to reach a specified tolerance level.

Upon execution of **Newton**, we are first asked to choose a mode. When we choose the Root Convergence mode and then press the "Go" button, the screen will look like Figure 1. (Be patient, there are a lot of "complex" calculations taking place to produce the graph in Figure 1).
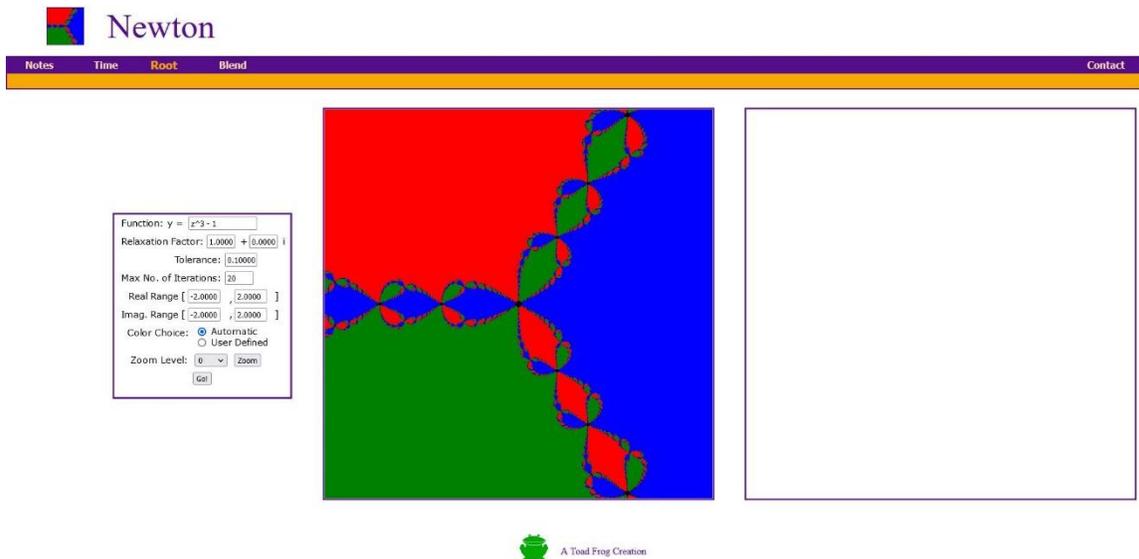

Figure 1

The function that is being used to produce the graph in Figure 1 is $F(z) = z^3 - 1$. Newton's method generates a sequence of (complex) numbers by starting with an initial value z_0 and successive iterations in the sequence are determined by the recursive formula $z_{n+1} = z_n - \frac{F(z_n)}{F'(z_n)}$, $n = 0, 1, 2, \cdots$. When Newton's method converges, it always converges to a zero of the function $F(z)$. Initially the graph window represents that portion of the complex plane for which the real part ranges from -2 to 2 and the imaginary part also ranges from -2 to 2. **Newton** will calculate the first few iterates for

the function $F(z)$ with initial value z_0 corresponding to the upper left corner of the graph window. If it appears that the Newton sequence will converge to a particular value, the upper left pixel is drawn in the color of the first zero of $F(z)$ found. The process continues, going down pixel by pixel. Each time the bottom of the graph window is reached the whole process is continued, but one pixel to the right. Also each time a new zero of $F(z)$ is approximated, a new color is assigned.

The zeros of the function $F(z) = z^3 - 1$ are the "cube roots of unity",
$$\cos\left(\frac{2k\pi}{3}\right) + \sin\left(\frac{2k\pi}{3}\right)i, \qquad k = 0, 1, 2$$

When $k = 0$, the cube root of unity is just the real number 1, and whenever Newton's method converges to 1, the pixel corresponding to the initial value used is colored blue. When $k = 1$, the cube root of unity is the complex number $\cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)i = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and whenever Newton's method converges to this root of unity, the pixel corresponding to the initial value used is colored red. When $k = 2$, the cube root of unity is the complex number $\cos\left(\frac{4\pi}{3}\right) + \sin\left(\frac{4\pi}{3}\right)i = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, and whenever Newton's method converges to this root of unity, the pixel corresponding to the initial value used is colored green. The resulting graph is called a Newton fractal.

If we wish to change the function $F(z)$, we go to the status window at the left of the screen and type in a new formula in the text box labeled "Function:" Figure 2 shows the resulting Newton fractal when $F(z) = z^3 - 2z + 2$. The areas colored black are those points which correspond to initial values for which two successive values of the Newton sequence did not get to within a distance of the specified tolerance of .1 within the specified maximum number of iterations of 20. The tolerance level and the maximum number of iterations can be changed by editing their values in the status window.

A modification of Newton's iteration is to start with an initial value z_0 and then generate the sequence $z_{n+1} = z_n - a\frac{F(z_n)}{F'(z_n)}$, $n = 0, 1, 2, \cdots$, where the constant a is the "relaxation factor". Initially the relaxation factor is set to 1, but its real part and its imaginary part can be adjusted by editing their values in the status window. Figure 3 shows the fractal that is generated when Newton's method is applied to the function $F(z) = z^3 - 1$ with relaxation factor 2. The tolerance level was increased to 0.25 and the maximum number of iterations was increased to 60. Also the portion of the complex plane showing was edited: the real parts and the imaginary parts of the complex numbers can now range from -3 to 3. Be aware that this figure will take a few minutes to complete on many personal computers.

Near the bottom of the status window are the "Color Choice" buttons. If we choose the "User defined" button, every time a new approximate zero of $F(z)$ is found, we will be prompted to choose a color, as in Figure 4. Figure 5 displays the same Newton fractal as Figure 3, but with three different colors (yellow, green, cyan) chosen.

Function: y = [z^3 - 2z + 2]
Relaxation Factor: [1.0000] + [0.0000] i
Tolerance: [0.10000]
Max No. of Iterations: [20]
Real Range [-2.0000 , 2.0000]
Imag. Range [-2.0000 , 2.0000]
Color Choice: ⦿ Automatic
○ User Defined
Zoom Level: [0 ⌄] [Zoom]
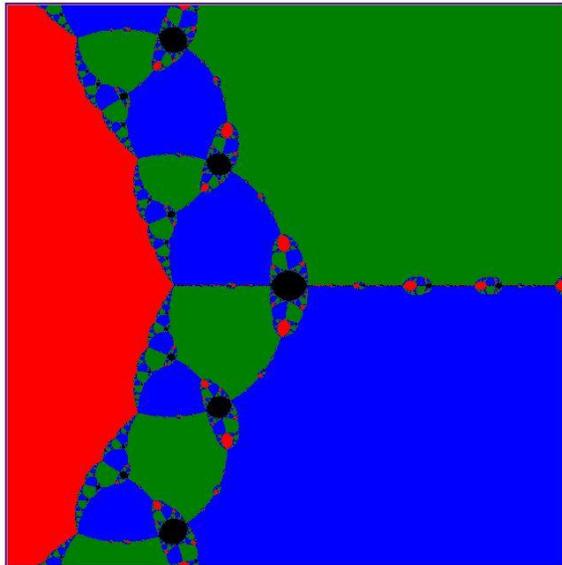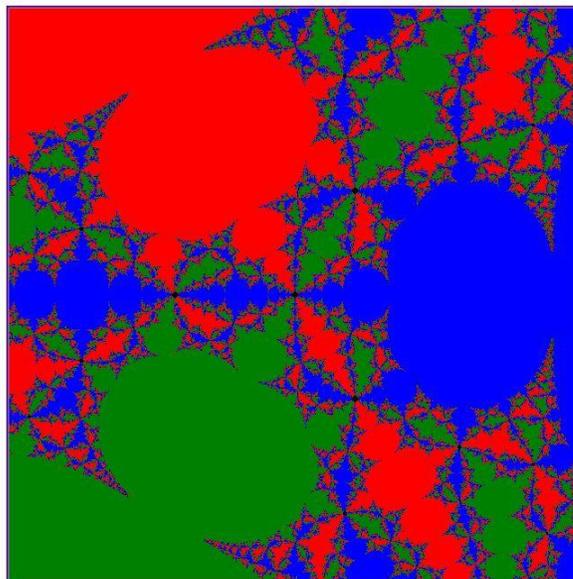[Go!]

Figure 2

Function: y = [z^3 - 1]
Relaxation Factor: [2.0000] + [0.0000] i
Tolerance: [0.25000]
Max No. of Iterations: [60]
Real Range [-3.0000 , 3.0000]
Imag. Range [-3.0000 , 3.0000]
Color Choice: ⦿ Automatic
○ User Defined
Zoom Level: [0 ⌄] [Zoom]
[Go!]

Figure 3

Move the '+' to choose color #1. Then press [OK]
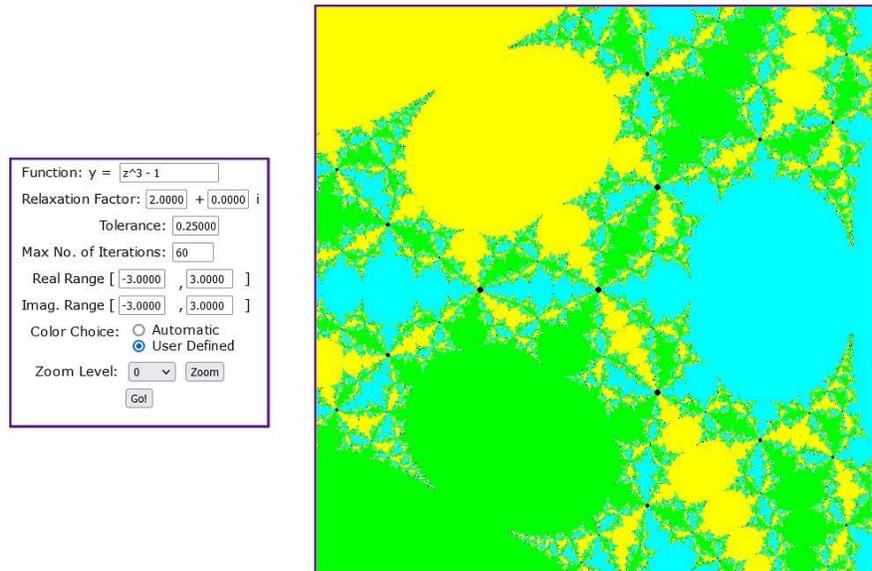
Figure 4

Figure 5

We can choose a different viewing window either by changing the numbers in the Real and Imaginary Range edit boxes or by pressing the Zoom button. When the Zoom box is pressed, we reposition the cursor to one corner of the zoom window, then press "OK". A "+" symbol appears at that location. (On ipads simply touch the location of the corner). Then we reposition the cursor to the diagonally opposite corner, the press "OK". Once the "Zoom It" button is pressed, the program automatically "squares" the zoom window, increments the zoom level by one, prints the new real and imaginary ranges in the status window, and graphs the zoomed portion of the Newton fractal in the right window. If the zoom level is zero, the middle window is used to define the new zoom window; otherwise the right window is used. We can go back to a previous zoom level by choosing its number in the zoom level menu. Figure 6 shows a zoomed portion of the fractal of Figure 5.
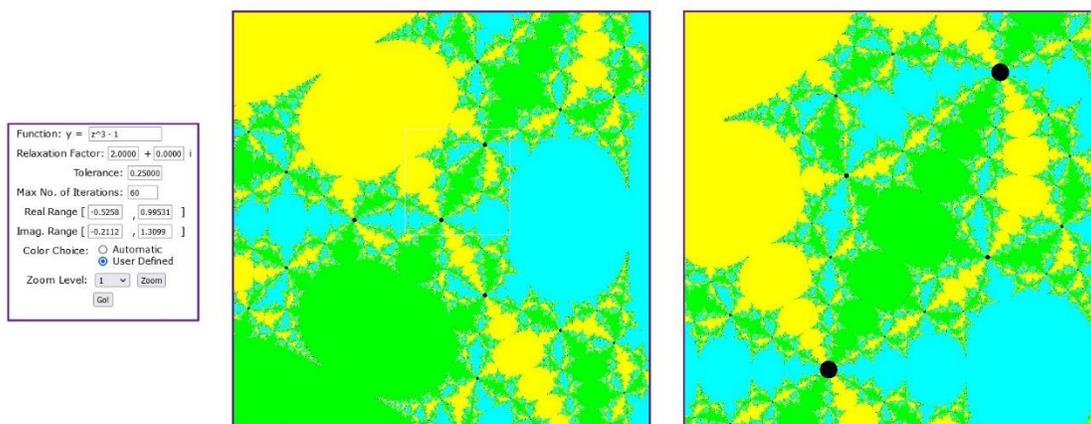

Figure 6

As a final example of the Root Convergence mode we consider the function $F(z) = z^8 + 15z^4 - 16$. Figure 7 shows the result along with a zoom in the upper right hand region. The maximum number of iterations was increased to 60.
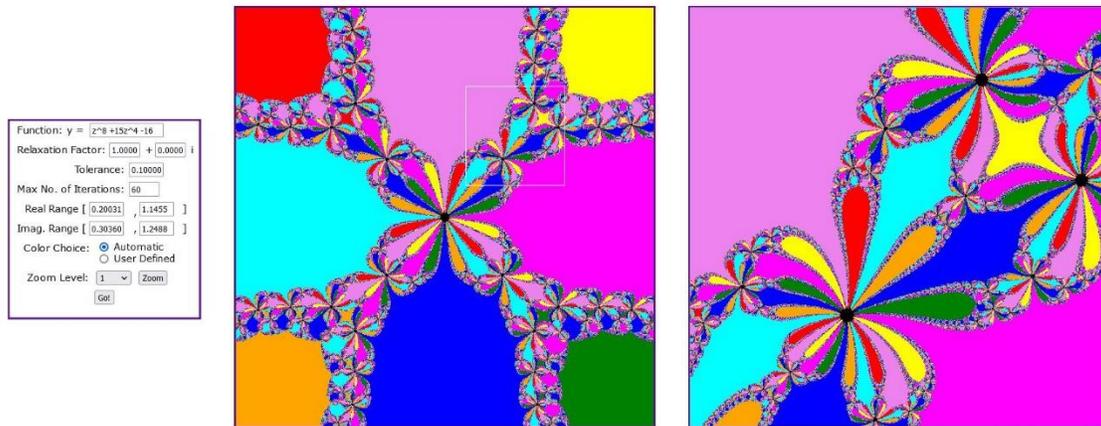

Figure 7

"Time to Converge" executes Newton's method on the specified function $F(z)$ at various initial values, but colors the pixel corresponding to an initial value according to how long it takes for two successive iterations to come to within the specified tolerance. In the status window the function, relaxation, tolerance, real range, and imaginary range text boxes work exactly like their counterparts in "Root Convergence". Initially, if the distance between the first iteration and the initial value is less than the specified tolerance level, the pixel corresponding to that initial value is colored cyan. If the distance between the first two iterations is less than the specified tolerance, the pixel corresponding to the initial value is colored blue. If the distance between the third and fourth iterations is less than the specified tolerance, the pixel corresponding to the initial value is colored magenta. If the distance between the fifth and sixth iterations is less than the specified tolerance, the pixel corresponding to the initial value is colored red. If the distance between the seventh and eighth iterations is less than the specified tolerance, the pixel corresponding to the initial value is colored forest green. If the distance between the ninth and tenth iterations is less than the specified tolerance, the pixel corresponding to the initial value is colored yellow. If the distance between none of the first ten successive iterations, is less than the specified tolerance level, then the pixel corresponding to that initial value is colored black. Figure 8 shows the fractal generated in this mode for the function $F(z) = z^3 - 1$.

The number of iterations required for a specified pixel color can be changed by editing the boxes under the heading "Iteration Color Limits" in the status window. Also if we click on one of the colored boxes, a prompt similar to that of Figure 4 appears. Once to "OK" button is pressed, the prompt window disappears and the appropriate box in the status window changes colors. Figure 9 shows the result of applying Newton's method to the function $F(z) = z^3 - 1$, but the iteration color limits of Figure 7 were all doubled and the color yellow was changed to gray.
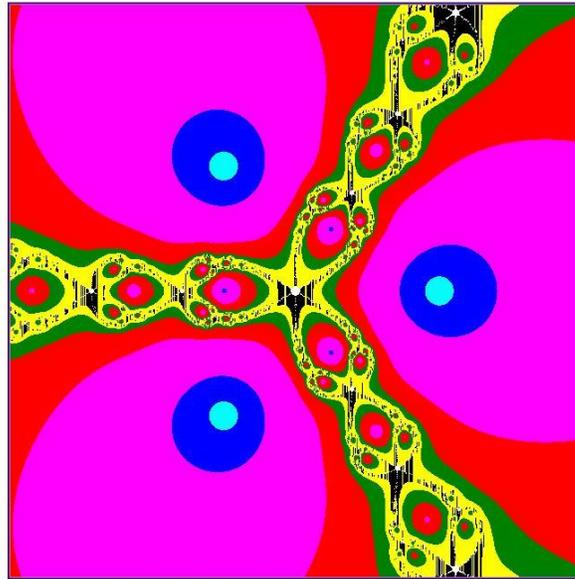
Figure 8

The Blend mode works much like the "Root Convergence" mode except that even though pixels are colored according to which zero of $F(z)$ Newton's method is converging, the colors are shaded darker the more iterations it takes for successive iterations to get within the specified tolerance level. The status window options work exactly like those in the Root Convergence mode. Figure 10 shows the result of applying Newton's method to the function $F(z) = z^5 - 3iz^3 - (5 + 2i)z^2 + 3z + 1$. The real and complex ranges were each extended to range from -4 to 4.
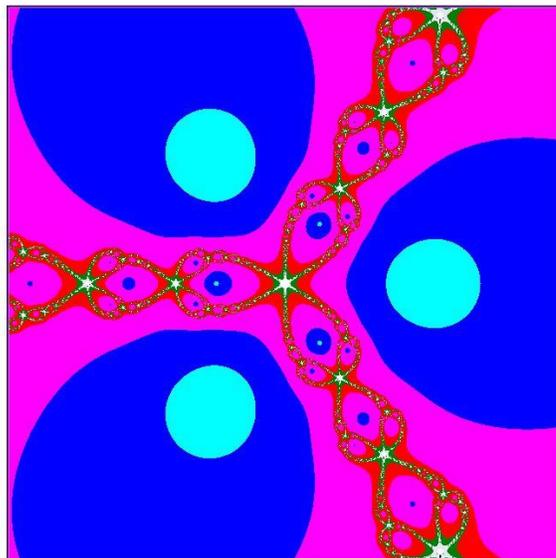
Figure 9

Function: y = [+2i)*z^2 + 3z + 1]
Relaxation Factor: [1.0000] + [0.0000] i
Tolerance: [0.10000]
Max No. of Iterations: [20]
Real Range [[-4.0000] , [4.0000]]
Imag. Range [[-4.0000] , [4.0000]]
Color Choice: ⊙ Automatic
○ User Defined
Zoom Level: [0 ⌄] [Zoom]
[Go!]

Figure 10