Examples 6.4 – Inverse Trigonometric Functions

1. Compute the following derivatives.

(a) $\frac{d}{dx}\left(\sin^{-1}\left(x^2\right)\right)$

(b) $\frac{d}{dx}\left(\ln\left(\tan^{-1}\left(\frac{1}{x}\right)\right)\right)$

Solution: (a) $\frac{d}{dx}\left(\sin^{-1}\left(x^2\right)\right) = \frac{1}{\sqrt{1-\left(x^2\right)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$

(b) $\frac{d}{dx}\left(\ln\left(\tan^{-1}\left(\frac{1}{x}\right)\right)\right) = \frac{1}{\tan^{-1}\left(\frac{1}{x}\right)} \cdot \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{-1}{x^2} = \frac{-1}{\left(1+x^2\right)\tan^{-1}\left(\frac{1}{x}\right)}$

2. Compute the following integrals.

(a) $\int \frac{-1}{\sqrt{1-(2x+1)^2}}dx$

(b) $\int \frac{5}{2+x^2}dx$

Solution: (a) Apply a u-substitution with $u = 2x+1$ to get

$$\int \frac{-1}{\sqrt{1-(2x+1)^2}}dx = \tfrac{1}{2}\cos^{-1}(2x+1) + C$$

(b) In order to use the arctangent formula, we need the form $1 + u^2$ in the denominator, not $2 + u^2$. However, we could divide the numerator and denominator by 2 to change the form. That is,

$$\int \frac{5}{2+x^2}dx = \int \frac{\frac{5}{2}}{1+\frac{x^2}{2}}dx = \frac{5}{2}\int \frac{1}{1+\left(\frac{x}{\sqrt{2}}\right)^2}dx$$

Finally, the substitution $u = \frac{x}{\sqrt{2}}$ should do the trick. In this case we will "pick up" a factor of $\frac{1}{1/\sqrt{2}} = \sqrt{2}$:

$$\int \frac{5}{2+x^2}dx = \frac{5\sqrt{2}}{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$