Activity 3.6‡ – Integrals of Polynomials

FOR DISCUSSION: In your own words, state each of the following:

- The power rule for derivatives and the power rule for integrals;
- The constant multiple and sum/difference rules for integrals;
- The Fundamental Theorem of Calculus for polynomials.

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1. Evaluate each of the following integrals. You may need to rewrite the integrand first.

(a) \[ \int (-2t^3 + 5t^2 + t - 4) \, dt = \]

(b) \[ \int \left( 3x^{\frac{3}{2}} - \frac{7}{x^2} \right) \, dx = \]

(c) \[ \int \sqrt{x} \, dx = \]

‡ This activity has supplemental exercises.
(d) \( \int \left( 3x + \frac{5}{\sqrt{x}} \right) dx = \)

(e) \( \int_{-1}^{1} (3u + 2)^2 \, du = \) (HINT: You must expand the integrand before you integrate.)

(f) \( \int_{1}^{4} \left( \frac{2x^3 - 32}{2x^3} \right) dx = \) (HINT: You must first split the integrand into two fractions.)
2. Recall that \( \int f(x) \, dx \) represents the infinite family of antiderivatives of \( f \), each identified by its constant of integration, \( C \). Given a point in the plane, we could find the constant \( C \) that identifies the unique member of the family passing through the given point.

Consider the function \( f(x) = \frac{7}{x^3} - \frac{6}{x^5} \), and suppose that \( F(x) \) is an antiderivative of \( f(x) \).

(a) Find a formula for \( F \) such that \( F(1) = 3 \). That is, find the antiderivative passing through the point \((1, 3)\).

(b) Find a formula for \( F \) such that \( F(2) = -\frac{41}{32} \). That is, find the antiderivative passing through the point \((2, -\frac{41}{32})\).
3. **(OPTIONAL)** Just as the derivative of a product is not the product of the derivatives, the integral of a product is not the product of the integrals. That is,
\[
\int (f(x) \cdot g(x)) \, dx \neq \int f(x) \, dx \cdot \int g(x) \, dx
\]
Later in this course and in Calculus II you will learn how to integrate certain types of products, but be aware that **there is not a “product rule” for integration!**

Think of two simple power functions \( f \) and \( g \) such that the integral of their product is not the product of their integrals. For simplicity, assume that all constants of integration are zero.

Let \( f(x) = \ldots \) and let \( g(x) = \ldots \).

The integral of the product is \( \int f(x) \cdot g(x) \, dx = \ldots \), but the product of the integrals is \( \int f(x) \, dx \cdot \int g(x) \, dx = \ldots \).

4. **(OPTIONAL)** Let’s verify the general properties of the integral.

(a) **Constant Multiple Rule:** Let \( F \) be a function such that \( F' = f \), and let \( k \) be a constant. For simplicity, assume that all constants of integration are zero.

(i) Since \( \frac{d}{dx} (k \cdot F(x)) = \ldots \), it follows that \( \int k \cdot f(x) \, dx = \ldots \).

(ii) Since \( \frac{d}{dx} (F(x)) = \ldots \), it follows that \( \int f(x) \, dx = \ldots \).

(iii) Put Parts (i) and (ii) together to deduce the constant multiple rule.

(b) **Sum/Difference Rule:** Let \( F \) and \( G \) be functions such that \( F' = f \) and \( G' = g \). For simplicity, assume that all constants of integration are zero.

(i) Since \( \frac{d}{dx} (F(x) \pm G(x)) = \ldots \), we have \( \int (f(x) \pm g(x)) \, dx = \ldots \).

(ii) Since \( \frac{d}{dx} (F(x)) = \ldots \), we have \( \int f(x) \, dx = \ldots \).

(iii) Since \( \frac{d}{dx} (G(x)) = \ldots \), we have \( \int g(x) \, dx = \ldots \).

(iv) Put Parts (i) through (iii) together to deduce the sum/difference rule.