Lesson 1.4 – Integrals of Constant Functions

Given a linear function \( y = mx + b \), we can find a unique constant slope (derivative) \( y' = m \). However, given a constant function \( y' = m \), there are an infinite number of lines having \( m \) as a slope. Lines that have the same slope \( m \) are vertical shifts of one another and have equations of the form \( y = mx + C \) for some constant \( C \). These lines form the family of antiderivatives of \( m \), which is denoted by the indefinite integral:

\[
\int m \, dx = mx + C \quad \text{(family of antiderivatives of } m) \tag{1}
\]

(The function between the integral sign \( \int \) and the differential \( dx \) is called the integrand)

Suppose we are given the rate of change \( f'(x) \) over some interval \([x_0, x_1]\) and we want the net (accumulated) change in \( f \) on \([x_0, x_1]\).

**Observation 1:** If \( f \) has a constant rate of change \( f'(x) = m \) on \([x_0, x_1]\), then \( f \) is an antiderivative of \( m \) of the form \( f(x) = mx + C \), for some constant \( C \). Note that the net change in \( f(x) = mx + C \) on \([x_0, x_1]\) for any \( C \) is

\[
f(x_1) - f(x_0) = (mx_1 + C) - (mx_0 + C) = mx_1 - mx_0
\]

**Observation 2:** Let \( A \) denote the net (signed) area bounded by \( f' \) from \( x_0 \) to \( x_1 \). From geometry,

\[
A = m(x_1 - x_0) = mx_1 - mx_0 = f(x_1) - f(x_0)
\]

The net (signed) area bounded by the constant function \( y = m \) on the interval \([x_0, x_1]\) is denoted by the

**Definite integral:** \[ \int_{x_0}^{x_1} m \, dx = \text{net signed area bounded by } m \text{ on } [x_0, x_1] \]

(Units: \( \text{units of } m \times \text{units of } x \))

**Fundamental Theorem of Calculus (for constant functions):** If \( f'(x) \) is a constant function, then the net area bounded by the graph of \( f'(x) \) on the interval \([x_0, x_1]\) is equal to the change in the (or any) antiderivative \( f(x) \) on \([x_0, x_1]\). That is,

\[
\int_{x_0}^{x_1} f'(x) \, dx = f(x)|_{x_0}^{x_1} = f(x_1) - f(x_0)
\]