AU

William Varick Nevins III High School Mathematics Competition
Fall 2006

Division of Mathematics and Computer Science
Alfred University
Alfred, NY 14802

Instructions:

1. This competition will last seventy-five minutes – from 10:05 to 11:20.

2. Put your five-digit student number in the correct place on the computer answer sheet.

3. The use of calculators is not permitted on this examination.

4. There are thirty questions. Mark your answers on the computer answer sheet. Use a #2 pencil only. You may use this question booklet for scratch work.

1. If $\frac{4^x}{2^{x-7}} = 8$ and $\frac{9^{x+4}}{3^{5y}} = 243$ then xy equals

   A) -4

   B) $\frac{12}{5}$

   C) 4

   D) 6

   E) 12

2. When the sides of a square are all reduced by three, the area is reduced by 45. The reduced area is

   A) 25

   B) 36

   C) 49

   D) 64

   E) 81

3. The shortest distance between the point (1, 2) and the line $x + y = 1$ is

   A) 1

   B) $\sqrt{2}$

   C) $\sqrt{3}$

   D) 2

   E) $\sqrt{5}$

4. At Juan's Pizzeria a medium pizza has a diameter of 10 inches. It has been said that the amount of pizza in one-third of the medium pizza is the same as the amount of pizza in one-quarter of the large pizza. If this is true, what is the diameter of the large pizza?

   A) $\frac{10\sqrt{3}}{3}$ in.

   B) $\frac{10\pi}{3}$ in.

   C) 11 in.

   D) $\frac{20\sqrt{3}}{3}$ in.

   E) 12 in.

5. If $x^3 = x$, then what is $x^6 + x^5 + x^4 + x^2 + 1$?

   A) 1

   B) 1 or 3

   C) 1 or 5

   D) 3 or 5

   E) 1, 3, or 5

6. If @ + # = 7 and @ + $ = 5, what does # – $ equal?

   A) 1

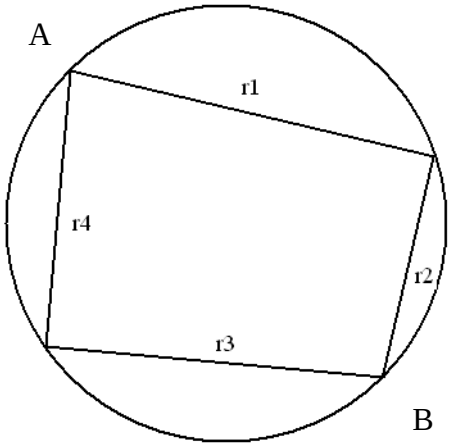
   B) 2

   C) 3

   D) 4

   E) 5

7. A quadrilateral is inscribed inside a circle such that the two vertices A and B lie on a diameter of the circle. We are given that r3 = $\sqrt{2}$ r1 and r4 = $\frac{1}{2}$ r1. What is the relationship between r2 and r1? Note that the accompanying figure is not drawn to scale.



A) r2 < r1

B) r2 = r1

C) r2 = $\sqrt{\frac{5}{4}}$ r1

D) r2 = $\frac{\pi}{8} r1$

E) r2 = $\frac{\pi^2}{9}$ r1

8. Students at AU print an average of 500 pieces of paper per student a semester. Unfortunately 5% of the students account for 30% of all paper printed. If these 5% are restricted to 500 pieces of paper and every other student continues to print as they have in the past, what will be the new average of paper printed per student per semester?

A) 75

B) 125

C) 250

D) 375

E) 500

9. From a group of boys and girls fifteen girls leave. Two boys remain for each girl. From this group, forty-five boys leave and five girls remain for each boy. How many girls were there in the original group?

   A) 40

   B) 50

   C) 60

   D) 65

   E) 75

10. The difference, sum, and product of two numbers are in the ratio 1 : 7 : 24. The larger of the two numbers is

    A) 4

    B) 8

    C) 14

    D) 42

    E) 56

11. If the line whose equation is $y = 4x + p$ intersects the parabola whose equation is $y = x^2 + x$ in exactly one point, then p is equal to

    A) $-\frac{9}{4}$

    B) 0

    C) $\frac{4}{9}$

    D) 1

    E) $\frac{9}{4}$

12. How many integers between 0 and 1,000 have exactly one digit equal to 5?

A) 200

B) 225

C) 226

D) 242

E) 243

13. A *magic square* is a 3 × 3 array in which the digits 1 through 9 appear exactly once, and the sum along each row, column, and diagonal is 15. For the 3 × 3 array below to be a magic square, the digit *x* must be

A) 1

B) 3

C) 5

D) 7

E) 9

a	b	c
d	x	e
f	g	h

14. An *arithmetic progression* is a sequence in which the difference between any two consecutive terms is constant. Suppose the sum of the 3^{rd} and 9^{th} terms of an arithmetic progression is 16. Then the sum of the first eleven terms of the progression is

A) 88

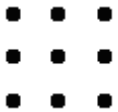
B) 98

C) 108

D) 118

E) 128

15. How many triangles can be drawn so that vertices of the triangle lie on the dots of the rectangle below?



A) 4

B) 9

C) 28

D) 75

E) 76

16. The C☺☺kie M☺nster has 15 jars of cookies containing 1,2,3,…,15 cookies respectively. He is allowed to choose any collection of jars and then he must remove the same number of cookies from each jar he has chosen; call such a choice and removal a "move". What is the least number of "moves" it will take to empty all the jars?

A) 1

B) 4

C) 5

D) 6

E) 15

17. A person begins with \$64 and successively bets one-half of her money. On each play she either gains that amount or loses that amount. How much money does she have after six plays if she wins three of those bets and loses three?

A) \$8

B) \$16

C) \$27

D) \$64

E) The answer depends on the order in which the wins and losses occur.

18. A farmer bought 749 sheep. He then sold 700 of them for the same total amount he paid for the 749 sheep. The remaining 49 sheep were sold at the same price per sheep as the other 700. The farmer's percent gain in the transaction is

    A) 6.5

    B) 6.75

    C) 7.0

    D) 7.5

    E) 8.0

19. Generate a sequence of numbers by starting with 1, 1, and then generating each successive number by adding the previous two. So the sequence begins $1, 1, 2, 3, 5, \ldots$ How many of the first 1000 numbers in this sequence are odd?

    A) 333

    B) 334

    C) 500

    D) 666

    E) 667

20. Given the sequence $10^{1/11}, 10^{2/11}, 10^{3/11}, ..., 10^{n/11}$, the smallest value of n such that the product of the first members of this sequence exceeds 100,000 is:

    A) 7

    B) 8

    C) 9

    D) 10

    E) 11

21. Let $f(x)=\left(1+\frac{1}{x}\right)\left(\frac{1}{x}+\frac{1}{x^2}\right)\left(\frac{1}{x^2}+\frac{1}{x^3}\right)$, $g(x)=\left(1-\frac{1}{x}\right)\left(\frac{1}{x}-\frac{1}{x^2}\right)\left(\frac{1}{x^2}-\frac{1}{x^3}\right)$, and $h(x)=x^{12}f(x)g(x)$. Find $h\left(\sqrt{3}\right)$.

A) $\sqrt{3}$

B) 3

C) 8

D) 27

E) 729

22. The 2006^{th} digit of the decimal expansion of $\frac{2}{7}$ is

A) 1

B) 2

C) 4

D) 5

E) 8

23. A goat is chained to a fence at a 90^0 corner where a 12 ft long and 26 ft. long fence meet. The 12ft long fence has another fence at a 60^0 angle to it that is also 12ft. long. What is the area of land that the goat can reach if the chain is 25ft long?

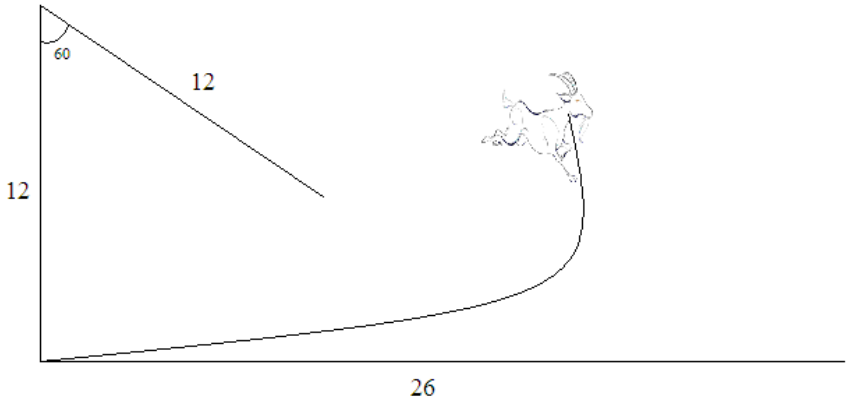
A) $156\ \text{ft}^2$

B) $100\frac{\sqrt{2}}{3}\ \text{ft}^2$

C) $\frac{435\pi}{4}+36\sqrt{3}\ \text{ft}^2$

D) $\frac{83\sqrt{29}\pi}{5}\ \text{ft}^2$

E) 144ft^2



24. A circle is inscribed in a square of side length 2 which in turn is inscribed in a circle. The area of the shaded region is

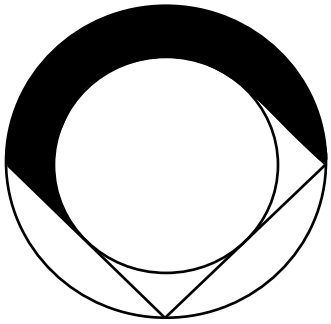
A) $\frac{\pi}{8}$

B) $\frac{\pi}{4}$

C) $\frac{\pi}{2}$

D) $\frac{2\pi}{3}$

E) $\frac{3\pi}{2}$



25. At his usual rate a man rows 15 miles downstream in five hours less than it takes him to return. If he doubles his usual rate, the time downstream is only one hour less than the time upstream. In miles per hour, the rate of the stream's current is:

A) 2

B) $\frac{5}{2}$

C) 3

D) $\frac{7}{2}$

E) 4

26. If angle *A* of the right triangle below is trisected, then the length of segment *w* in terms of the lengths of segments *x*, *y*, and *z* is

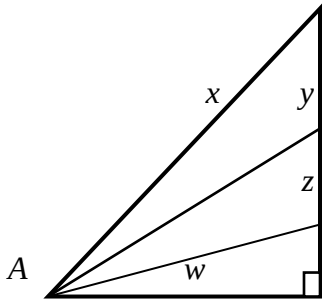
A) $\frac{x^2}{zy}$

B) $\frac{xz}{y}$

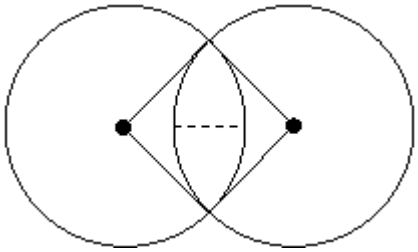
C) $\frac{x}{zy}$

D) $\frac{xy}{z}$

E) $\frac{yz}{x}$



27. Two circles centered at diagonal vertices of a 1 by 1 square intersect at the other two vertices, as shown below. Find the length of the dashed line.



A) 1

B) $2\sqrt{2}$

C) $2-2\sqrt{2}$

D) $2-\sqrt{2}$

E) $2+\sqrt{2}$

28. A pyramid has a square base with side length 2 and height 1. A cube is placed inside the pyramid so that it sits on the square base and its top corners each touch the slanted edges of the pyramid. The volume of the cube is:

A) $\frac{1}{64}$

B) $\frac{1}{27}$

C) $\frac{1}{8}$

D) $\frac{8}{27}$

E) $\frac{27}{64}$

29. A pile of gold and silver coins contains more silver coins than gold, but the total weight of each type is the same. When enough silver coins are removed to make the number of silver and gold coins equal, the removed silver coins weigh one-third of the weight of the remaining gold and silver coins. What is the ratio of the weight of a silver coin to a gold coin?

A) 1 to 2

B) 1 to 3

C) 2 to 3

D) 2 to 5

E) 3 to 4

30. A man on his way to dinner shortly after 6:00 p.m. observes that the hands of his watch form an angle of 110^0. Returning before 7:00 p.m., he notices that again the hands of his watch form an angle of 110^0. The number of minutes he has been away is:

A) $36\frac{2}{3}$

B) 40

C) 42

D) $42\frac{2}{5}$

E) 45