Lesson 8.6 – Integration by Substitution

Previously we learned that a reverse of the chain rule allows us to evaluate integrals such as $\int e^{-x}dx$, $\int \cos(\pi x)dx$, and $\int \frac{1}{1+(7x)^2}dx$. In fact, any integral of the form $\int f(kx)dx$ can be evaluated this way as long as we know an antiderivative F of f (see Activity 8.6). We generalize this idea in this lesson.

According to the chain rule, $\frac{d}{dx}\left(F(g(x))+C\right) = f(g(x))\cdot g'(x)$, where $f = F'$. By reversing this process, we are able to integrate general functions of the form $f(g(x))\cdot g'(x)$. That is,

$$\int f(g(x))\cdot g'(x)\,dx = F(g(x))+C.$$

***u*-substitution for indefinite integrals:**

- Look for a composite $f(g(x))$ in the integrand for which the antiderivative of f is known.
- Let $u = g(x)$ ("inside" exponent, radical, parentheses, denominator) so that $du = g'(x)dx$.
- Substitute u and du to get a new integral in terms of u. This may not always be possible!
- Integrate the new integral. This may not always be possible!
- Replace u with $g(x)$ in the final answer.
- Check your answer by differentiation.

***u*-substitution for definite integrals**: When we apply substitution to a definite integral defined along the x-axis, we turn it into one defined along the new u-axis. Therefore, the old x-limits are not valid for the new integral. One method is to convert both the integrand *and* limits from x's to u's, and then evaluate the definite integral directly over the u-axis using the u-limits. The other method is to convert the integrand from x's to u's, evaluate the indefinite integral over the u-axis, convert it back to the x-axis, and evaluate the definite integral over the x-axis using the x-limits. Here is a summary of the methods:

Method 1: Apply u-substitution, and at the same time convert x-limits of integration to u-limits of integration. Formally,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du = F(u)\Big|_{u=g(a)}^{u=g(b)} = F(g(b)) - F(g(a))$$

Method 2: First apply u-substitution to get $\int f(g(x))g'(x)\,dx = F(g(x))+C$. Then evaluate the right-hand side at the x-limits. That is,

$$\int_a^b f(g(x))g'(x)dx = F(g(x))\Big|_{x=a}^{x=b} = F(g(b)) - F(g(a))$$