Examples 8.6 – Integration by Substitution

1. Evaluate $\int 3x^2 \cos(x^3)\,dx$.

   Solution: Note that $\cos(x^3)$ is a composite function. Let $u = x^3$ (inside parentheses) so that $du = 3x^2 dx$. After substituting, the integral becomes

$$\int 3x^2 \cos(x^3)\,dx = \int \cos(x^3) \cdot 3x^2 dx = \int \cos u\,du = \sin u + C = \sin(x^3) + C$$

2. Evaluate $\int 9x^4 \sqrt{1+4x^5}\,dx$.

   Solution: Note that $\sqrt{1+4x^5}$ is a composite function. Let $u = 1+4x^5$ (inside radical) so that $du = 20x^4 dx$ and $x^4 dx = \frac{1}{20} du$. The substitution yields

$$\int 9x^4 \sqrt{1+4x^5}\,dx = 9\int \sqrt{1+4x^5} \cdot x^4 dx = \tfrac{9}{20}\int \sqrt{u}\,du = \tfrac{9}{20} \cdot \tfrac{2}{3} u^{3/2} + C = \tfrac{3}{10}\left(1+4x^5\right)^{3/2} + C$$

3. Evaluate the definite integral $\int_0^3 \frac{x+1}{x^2+2x+2}\,dx$.

   Solution: We will demonstrate both methods.

   Method 1: Let $u = x^2+2x+2$ (inside denominator) so that $du = (2x+2)dx = 2(x+1)dx$ and $(x+1)dx = \frac{1}{2} du$. Change the x-limits to u-limits by plugging in $x = 0$ and $x = 3$ into $u = x^2+2x+2$ to get $u = 2$ and $u = 17$, respectively. Hence,

$$\int_0^3 \frac{x+1}{x^2+2x+2}\,dx = \int_0^3 \frac{1}{x^2+2x+2}(x+1)dx = \tfrac{1}{2}\int_2^{17} \frac{1}{u}\,du = \tfrac{1}{2}\ln|u|\Big|_2^{17} = \tfrac{1}{2}(\ln 17 - \ln 2)$$

   Method 2: First, we will find an antiderivative of the integrand. Let $u = x^2+2x+2$ (inside denominator) so that $du = (2x+2)dx = 2(x+1)dx$ and $(x+1)dx = \frac{1}{2} du$. We have

$$\int \frac{x+1}{x^2+2x+2}\,dx = \int \frac{1}{x^2+2x+2}(x+1)dx = \tfrac{1}{2}\int \frac{1}{u}\,du = \tfrac{1}{2}\ln|u| + C = \tfrac{1}{2}\ln|x^2+2x+2| + C$$

   Therefore,

$$\int_0^3 \frac{x+1}{x^2+2x+2}\,dx = \tfrac{1}{2}\ln|x^2+2x+2|\Big|_0^3 = \tfrac{1}{2}(\ln 17 - \ln 2)$$