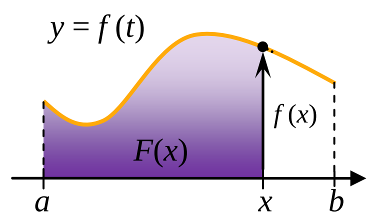
Lesson 8.5 – The Fundamental Theorem of Calculus (Part 2)

Part 2 of the FTC says that every continuous function has an antiderivative.

Fundamental Theorem of Calculus, Part 2: If f is continuous on $[a, b]$, then the function defined by $F(x) = \int_a^x f(t)dt$ for all x in $[a, b]$ is an antiderivative of f on $[a, b]$. That is,

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\int_a^x f(t)dx\right) = f(x)$$



Proof: The function $F(x) = \int_a^x f(t)dt$ is the net area bounded by f and the t-axis on $[a, x]$. We want to show that $F' = f$, in which case F is an antiderivative of f. By the definitions of the derivative of F, and of F itself, we have

$$\begin{aligned}
F'(x) &= \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(F(x+\Delta x) - F(x)\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\int_a^{x+\Delta x} f(t)dt + \int_x^a f(t)dt\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\int_x^{x+\Delta x} f(t)dt\right)
\end{aligned}$$

Since f is continuous on $[x, x+\Delta x]$, the MVT for integrals (Activity 8.4) implies that there is a point t^* in $[x, x+\Delta x]$ such that $\int_x^{x+\Delta x} f(t)dt = f(t^*)\left((x+\Delta x) - x\right) = f(t^*)\Delta x$. Therefore,

$$F'(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(f(t^*)\Delta x\right) = \lim_{\Delta x \to 0} f(t^*)$$

Now t^* is in the interval $[x, x+\Delta x]$, so as $\Delta x \to 0$, $t^* \to x$, and it follows that $f(t^*) \to f(x)$ by continuity. We have shown that $F'(x) = f(x)$, and the proof is complete. ■

If F is a composite, say $F(g(x)) = \int_a^{g(x)} f(t)\,dt$, then the chain rule applies and Part 2 becomes

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)dt\right) = f(g(x)) \cdot g'(x)$$