Lesson 8.4 – The Fundamental Theorem of Calculus (Part 1)

The Fundamental Theorem technically has two parts. Loosely speaking, the first part describes how to find the definite integral of a derivative, and the second part describes how to differentiate a function that is expressed as an integral. Along the way, we have discussed and used the first part of Fundamental Theorem of Calculus in the context of certain types of functions. We are now prepared to verify it for *any* continuous function.

Fundamental Theorem of Calculus, Part 1: If f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = \int_a^b F'(x)dx = F(x)|_a^b = F(b) = F(a)$$

Proof with subintervals of equal width: Let $x_0 = a$ and $x_n = b$ and divide $[x_0, x_n]$ into n subintervals of equal width, namely $[x_0, x_1]$, $[x_1, x_2]$, … , $[x_{k-1}, x_k]$, … , $[x_{n-1}, x_n]$. Since F is an antiderivative of f on $[a, b]$, F is differentiable and continuous on each $[x_{k-1}, x_k]$, and so F satisfies the hypothesis of the Mean Value Theorem on each $[x_{k-1}, x_k]$. It follows that there exists an x_k^* in $[x_{k-1}, x_k]$ such that

$$f(x_k^*) = F'(x_k^*) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = \frac{F(x_k) - F(x_{k-1})}{\Delta x}$$

and

$$f(x_k^*)\Delta x = F(x_k) - F(x_{k-1})$$

so that

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left(F(x_k) - F(x_{k-1})\right) = \sum_{k=1}^{n} F(x_k) - \sum_{k=1}^{n} F(x_{k-1})$$

The right-hand side is the telescoping sum from Activity 8.1 and is equal to

$$F(x_n) - F(x_0) = F(b) - F(a)$$

hence

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = F(b) - F(a)$$

Finally, as $n \to +\infty$, the right-hand side is constant, and the left-hand side becomes the definite integral. Therefore,

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \blacksquare$$