Lesson 8.3 – Rolle's Theorem and the Mean Value Theorem

Our goal over the next three lessons is to verify the FTC for any continuous function. The first step is to consider the formal definitions of continuity and differentiability on a closed interval, and to state two important theorems:

A function f is **continuous on [a, b]** if its graph has no breaks, jumps, or holes in [a, b]:

1. f is continuous at each c in (a, b): $\quad \lim\limits_{x \to c \in (a,b)} f(x) = f(c)$
2. f is right-continuous at a: $\quad \lim\limits_{x \to a^+} f(x) = f(a)$
3. f is left-continuous at b: $\quad \lim\limits_{x \to b^-} f(x) = f(b)$

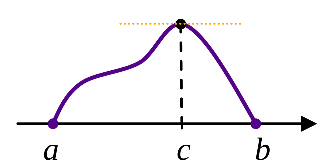
A function f is **differentiable on [a, b]** if its graph has no breaks, jumps, holes, corners, or vertical tangents in [a, b]:

1. f is differentiable at each c in (a, b): $\quad \lim\limits_{x \to c \in (a,b)} \frac{f(x) - f(c)}{x - c}$ exists
2. f is right-differentiable at a: $\quad \lim\limits_{x \to a^+} \frac{f(x) - f(a)}{x - a}$ exists
3. f is left- differentiable at b: $\quad \lim\limits_{x \to b^-} \frac{f(x) - f(b)}{x - b}$ exists

Note: In the first example from Lesson 4.3, we showed that if a function is differentiable, then it must be continuous. We also showed that a continuous function need not be differentiable.

Leaving home and then returning some time later requires stopping instantaneously and turning around (in rectilinear motion, of course). Therefore, there must be at least one point on the position curve at which the velocity is zero. This observation is the basis of Rolle's Theorem.

Rolle's Theorem: If f is continuous on [a, b], differentiable on (a, b), and $f(a) = f(b) = 0$, then there is at least one point c in (a, b) such that $f'(c) = 0$.



The Mean Value Theorem is the generalization of Rolle's Theorem. It says that there exists a tangent line parallel to the secant line as long as we have continuity and differentiability. In terms of rectilinear motion, there exists an instantaneous rate that equals the average rate.

Mean Value Theorem: If f is continuous on [a, b] and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$