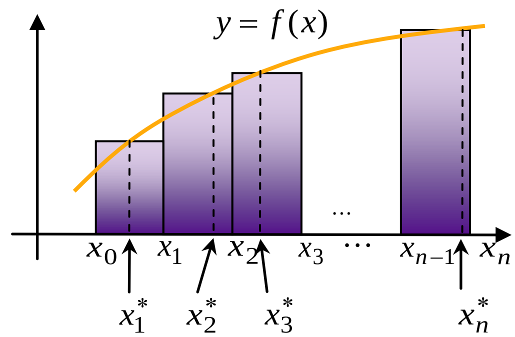
Lesson 8.2 – The Definition of Net Area

In this lesson, we formalize the definitions of net area and the definite integral.



Suppose we want the net area bounded by the graph of the continuous function f on the closed interval $[a, b]$.

STEP 1: Let $x_0 = a$ and $x_n = b$ and divide $[x_0, x_n]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$.

STEP 2: For each $k \geq 1$, choose *any* point x_k^* in the kth subinterval $[x_{k-1}, x_k]$. Common choices include left-hand endpoints, midpoints, and right-hand endpoints:

LH: $x_k^* = a + (k-1)\Delta x$ Mid: $x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x$ RH: $x_k^* = a + k \cdot \Delta x$

STEP 3: For each $k \geq 1$, construct the kth rectangle of height $f(x_k^*)$ and width Δx:

$$\text{Area of } k\text{th rectangle} = f(x_k^*)\Delta x$$

STEP 4: Add up the areas of the n rectangles to get an approximation A_n of the net signed area:

$$A_n = \sum_{k=1}^{n} f(x_k^*)\Delta x = \left(f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)\right)\Delta x \qquad \text{(a } \textbf{Riemann sum}\text{)}$$

STEP 5: As $n \to +\infty$ ($\Delta x \to 0$), the approximations A_n approach the **net signed area** $\int_a^b f(x)dx$:

Limit definition of the definite integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\frac{b-a}{n}$$

A function f is said to be **integrable** on $[a, b]$ if $\lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$ exists and does not depend on the subintervals (they need not even be of equal width) or on the choices for the x_k^*.

Formal Properties: Assume f and g are integrable on $[a, b]$, k is a constant, and c is in $[a, b]$.

1. $\int_a^a f(x)dx = 0$
2. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
3. $\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f(x)dx$
4. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^a f(x)dx$