Examples 8.2 – The Definition of Net Area

1. Evaluate $\int_{-1}^{2}\left(x^2-3x+1\right)dx$ using the limit definition of the definite integral.

Solution: In the given integral $a = -1$ and $b = 2$. Therefore, $\Delta x = \frac{b-a}{n} = \frac{(2)-(-1)}{n} = \frac{3}{n}$. For right-hand endpoints, $x_k^* = a + \Delta x \cdot k = -1 + \frac{3}{n}\cdot k$, hence

$$f\left(x_k^*\right) = \left(-1+\frac{3}{n}\cdot k\right)^2 - 3\left(-1+\frac{3}{n}\cdot k\right)+1 = 5-\frac{15}{n}\cdot k+\frac{9}{n^2}\cdot k^2,$$

$$f\left(x_k^*\right)\Delta x = \left(5-\frac{15}{n}\cdot k+\frac{9}{n^2}\cdot k^2\right)\left(\frac{3}{n}\right) = \frac{15}{n}-\frac{45}{n^2}\cdot k+\frac{27}{n^3}\cdot k^2$$

and

$$\begin{aligned}\sum_{k=1}^{n} f\left(x_k^*\right)\Delta x &= \frac{15}{n}\sum_{k=1}^{n}1-\frac{45}{n^2}\sum_{k=1}^{n}k+\frac{27}{n^3}\sum_{k=1}^{n}k^2\\ &= \frac{15}{n}(n)-\frac{45}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\\ &= 15-\frac{45}{2}\left(\frac{n(n+1)}{n^2}\right)+\frac{9}{2}\left(\frac{n(n+1)(2n+1)}{n^3}\right)\end{aligned}$$

By the definition of the definite integral,

$$\int_{-1}^{2}\left(x^2-3x+1\right)dx = \lim_{n\to+\infty}\left(15-\frac{45}{2}\left(\frac{n(n+1)}{n^2}\right)+\frac{9}{2}\left(\frac{n(n+1)(2n+1)}{n^3}\right)\right) = 15-\frac{45}{2}+9=\frac{3}{2}$$

2. Use the properties of the definite integral to evaluate $\int_1^4 |x-2|\,dx$.

Solution: Since an absolute value function is piecewise defined, we can use property 5 from Lesson 8.2 to integrate each piece. In particular, since

$$|x-2| = \begin{cases} -x+2 & \text{if } x<2 \\ x-2 & \text{if } x\geq 2\end{cases}$$

we have

$$\int_1^4 |x-2|\,dx = \int_1^2 |x-2|\,dx + \int_2^4 |x-2|\,dx$$

Therefore,

$$\int_1^4 |x-2|\,dx = \int_1^2(-x+2)dx+\int_2^4(x-2)dx = \left(-\tfrac{1}{2}x^2+2x\right)\Big|_1^2+\left(\tfrac{1}{2}x^2-2x\right)\Big|_2^4 = \tfrac{5}{2}$$