Examples 8.1 – Sigma Notation and Summations

1. Find the sum by expanding and adding.

(a) $\sum_{k=3}^{7} k$ (b) $\sum_{k=-2}^{3} k^3$

Solution: (a) $\sum_{k=3}^{7} k = 3+4+5+6+7 = 25$

(b) $\sum_{k=-2}^{3} k^3 = (-2)^3 + (-1)^3 + (0)^3 + (1)^3 + (2)^3 + (3)^3 = 27$

2. Find the sum by using the closed form summation formulas.

(a) $\sum_{k=1}^{50} 1$ (b) $\sum_{k=1}^{1000} k$ (c) $\sum_{k=3}^{99} k^2$ (d) $\sum_{k=-2}^{25} k^3$

Solution: (a) $\sum_{k=1}^{50} 1 = 50$

(b) $\sum_{k=1}^{1000} k = \dfrac{1000 \cdot 1001}{2} = 500{,}500$

(c) $\sum_{k=3}^{99} k^2 = \left(\sum_{k=1}^{99} k^2 \right) - 1^2 - 2^2 = \dfrac{99 \cdot 100 \cdot (2 \cdot 99 + 1)}{6} - 5 = 328{,}345$

(d) $\sum_{k=-2}^{25} k^3 = (-2)^3 + (-1)^3 + (0)^3 + \left(\sum_{k=1}^{25} k^3 \right) = -9 + \dfrac{25^2 \cdot 26^2}{4} = 105{,}616$

3. Write the summation $\sum_{k=1}^{n} \dfrac{1-2k+4k^3}{n^4}$ in closed form. Assume that n is a positive integer.

Solution:
$$\sum_{k=1}^{n} \frac{1-2k+4k^3}{n^4} = \frac{1}{n^4}\left(\sum_{k=1}^{n} 1 - 2\sum_{k=1}^{n} k + 4\sum_{k=1}^{n} k^3 \right)$$
$$= \frac{1}{n^4}\left(n - 2 \cdot \frac{n(n+1)}{2} + 4 \cdot \frac{n^2(n+1)^2}{4} \right)$$
$$= \frac{1}{n^4}\left(n - n^2 - n + n^4 + 2n^3 + n^2 \right)$$
$$= \frac{n+2}{n}$$

Note that this sum is finite for all n. In fact, $\lim_{n \to +\infty} \sum_{k=1}^{n} \dfrac{1-2k+4k^3}{n^4} = \lim_{n \to +\infty} \dfrac{n+2}{n} = 1.$