Lesson 7.5 – Differential Equations

Loosely speaking, a **differential equation** is an equation that contains one or more derivatives of a function *y*. To solve a differential equation means to find all functions *y* that satisfy the equation. If $y = y(t)$, then the equations $y' = 10t$ and $y' = 10y$ are both examples of differential equations. The first equation can be solved for *y* simply by integrating both sides with respect to *t*. That is,

$$y' = 10t \quad \rightarrow \quad y = 5t^2 + C$$

An equation like $y' = 10y$ cannot be solved by direct integration since *y* is unknown. That is,

$$y' = 10y \quad \rightarrow \quad y = 10\int y(t)dt = ???$$

Some students are required to take an entire course in differential equations, but for now we will consider only a few basic forms. You will verify these results in Activity 7.5.

In some situations (e.g., population) the rate at which a quantity grows or decays is proportional to the quantity itself. In general, which function *y* satisfies $y' = \pm ky$ $(k > 0)$? Clearly, the answer must be exponential:

$$y' = \pm ky \quad \rightarrow \quad y = Ce^{\pm kt} = y(0)e^{\pm kt}$$

This result is at the heart of **Newton's Law of Cooling** (see Activity 5.4). It states that the rate at which an object cools is proportional to the difference between the temperature *T* of the object and the constant ambient temperature T_a. That is, there exists a constant $k > 0$ such that

$$T'(t) = -k\left(T(t) - T_a\right)$$

If we let $y = T(t) - T_a$ and $T_0 = T(0)$, then $y' = T'(t)$ and $y(0) = T_0 - T_a$. Therefore, Newton's Law of Cooling has the form $y' = -ky$. From above, the solution is $y = y(0)e^{-kt}$, and so $T(t) - T_a = y(0)e^{-kt} = (T_0 - T_a)e^{-kt}$. Hence the temperature of the object at time *t* is given by

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

Sometimes the rate at which the *rate of change* of a quantity grows or decays is proportional to the quantity itself. For instance, since a spring tends to return to its equilibrium position, the acceleration of a spring is in the opposite direction of the motion of an attached mass. In general, which function *y* satisfies the differential equation $y'' = \pm ky$? In this case, the answer depends on whether the right-hand side is positive or negative:

$$y'' = ky \ \ (k > 0) \quad \rightarrow \quad y = C_1 e^{\sqrt{k}\,t} + C_2 e^{-\sqrt{k}\,t}$$

$$y'' = -ky \ \ (k > 0) \quad \rightarrow \quad y = C_1 \cos\left(\sqrt{k}\,t\right) + C_2 \sin\left(\sqrt{k}\,t\right)$$