Examples 7.5 – Differential Equations

1. A cup of coffee at 200°F is allowed to cool in a room with a constant temperature of 75°F. If the temperature of the coffee is 185°F after 3 min, what will the temperature be after 10 min?

   Solution: According to the given conditions and Newton's Law of Cooling, the temperature at time t is

   $$T(t) = T_a + (T_0 - T_a)e^{kt} = 75 + (200 - 75)e^{-kt} = 75 + 125e^{-kt}$$

   We are given that $T(3) = 75 + 125e^{-k(3)} = 185$, so $e^{-3k} = \frac{110}{125}$, and $-3k = \ln\left(\frac{110}{125}\right)$. It follows that $k = -\frac{1}{3}\ln\left(\frac{110}{125}\right) \approx 0.042611$. Therefore, after 10 minutes the temperature of the coffee will be $T(10) = 75 + 125e^{-0.042611\cdot(10)} \approx 156.6$ °F.

2. Find the general solution to the differential equation $y'' = 9y$, and then find the constants C_1 and C_2 in the general solution given that $y(0) = 1$ and $y'(0) = 15$.

   Solution: The general solution is $y = C_1e^{\sqrt{9}\,t} + C_2e^{-\sqrt{9}\,t} = C_1e^{3t} + C_2e^{-3t}$, and the rate of change is $y' = 3C_1e^{3t} - 3C_2e^{-3t}$. The condition $y(0) = 1$ implies that $C_1 + C_2 = 1$, and the condition $y'(0) = 15$ implies that $3C_1 - 3C_2 = 15$. Adding three times the first equation to the second equation eliminates the C_2 terms and we find that $6C_1 = 18$, or $C_1 = 3$. Plugging this value back into the first equation gives $C_2 = -2$. Hence, the solution is $y = 3e^{3t} - 2e^{-3t}$.

3. A mass attached to a vertical spring has position $y(t)$ meters after t seconds, where y satisfies $y'' = -4y$. Positions below equilibrium and downward motion are considered positive.
   (a) Find the position function if the initial position is 0.5 m and the initial velocity is 3 m/s.
   (b) Find a time at which the mass is at its greatest distance from equilibrium. Use this answer to find the amplitude of the system.

   Solution: (a) The general solution to $y'' = -4y$ is $y(t) = C_1\cos(2t) + C_2\sin(2t)$, and so the velocity function is $y'(t) = -2C_1\sin(2t) + 2C_2\cos(2t)$ m/s. The initial conditions yield $y(0) = C_1 = 0.5$ and $y'(0) = 2C_2 = 3$, hence the position function satisfying the given initial conditions is $y(t) = 0.5\cos(2t) + 1.5\sin(2t)$ m.

   (b) The greatest distance occurs when the velocity $y'(t) = -\sin(2t) + 3\cos(2t) = 0$. By adding $\sin(2t)$ to both sides and then dividing by $\cos(2t)$, we can rewrite the equation in terms of the tangent function as $3 = \frac{\sin(2t)}{\cos(2t)} = \tan(2t)$. Therefore, $t = \frac{1}{2}\tan^{-1}(3) \approx 0.625$ s, and it follows that the amplitude is $y\left(\frac{1}{2}\tan^{-1}(3)\right) = 0.5\cos\left(2\cdot\frac{1}{2}\tan^{-1}(3)\right) + 1.5\sin\left(2\cdot\frac{1}{2}\tan^{-1}(3)\right) \approx 1.58$ m.