Activity 7.5†‡ – Differential Equations

FOR DISCUSSION: *Which function has a first derivative proportional to itself?*
Which function has a second derivative proportional to itself?
Which function has a second derivative proportional to its negative?

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1. Write down the general solution to each differential equation.

   (a) $y' = 5y \quad \rightarrow \quad y =$

   (b) $y' = -3y \quad \rightarrow \quad y =$

   (c) $y'' = 25y \quad \rightarrow \quad y =$

   (d) $y'' = -15y \quad \rightarrow \quad y =$

2. A cup of coffee at 185°F is allowed to cool in a room with a constant temperature of 70°F. If the temperature of the coffee is 170°F after 5 minutes, what will the temperature be after 15 minutes?

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† This activity is referenced in Lesson 7.5.
‡ This activity has supplemental exercises.

3. Find the general solution to the differential equation $y'' - 16y = 0$, and then find the constants C_1 and C_2 in the general solution given that $y(0) = 3$ and $y'(0) = -2$.

4. A mass attached to a vertical spring has position $y(t)$ inches after t seconds, where y satisfies $4y'' + y = 0$. Positions below equilibrium and downward motion are considered positive. Find the position function if the initial position is 1.5 inches above equilibrium and the initial velocity is 5 in/s.

5. Suppose that $y = 10e^{-7t}$, and notice that $y' = -7\left(10e^{-7t}\right) = -7y$. This computation shows that $y = 10e^{-7t}$ is a solution to the differential equation $y' = -7y$.

   Perform a similar computation to verify that $y = C_0 e^{\pm kt}$ is a solution to the differential equation $y' = \pm ky$ for $k > 0$.

6. Verify that $y = C_1 e^{\sqrt{k}\,t} + C_2 e^{-\sqrt{k}\,t}$ is a solution to the differential equation $y'' = ky$ for $k > 0$.

$y' =$

$y'' =$

$ky =$

7. Verify that $y = C_1 \cos\left(\sqrt{k}\,t\right) + C_2 \sin\left(\sqrt{k}\,t\right)$ is a solution to the equation $y'' = -ky$ for $k > 0$.

$y' =$

$y'' =$

$-ky =$

8. (**OPTIONAL**) In Part 5, we merely verified that any function of the form $y = C_0 e^{\pm kt}$ is a solution to the differential equation $y' = \pm ky$. We will now prove that these are the *only* solutions. First we write $y' = \pm ky$ in Leibniz notation as $\frac{dy}{dt} = \pm ky$. Then we "separate" the t's and y's to get

$$\frac{1}{y} dy = \pm k dt$$

(a) Integrate the left-hand side with respect to y, and integrate the right-hand side with respect to t. This will introduce a logarithm on the left-hand side. You need only include one constant of integration C, so place it on the right-hand side.

(b) Solve the result from Part (a) for y. Try to identify C_0.