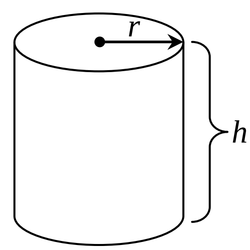
Lesson 7.4 – The Extreme Value Theorem and Optimization

We use the terms *local maxima* and *local minima* to refer to all of the peaks and valleys of a graph on an interval. We use the terms **absolute (or global) maximum** and **absolute (or global) minimum** to refer to the unique largest and smallest values, respectively, of a graph on an interval. In other words, the absolute extrema are the largest and smallest range values that the function attains on a given interval. The existence of absolute extrema may depend on the interval in question and also on continuity. Thus, a function may have local extrema on an interval without having any absolute extrema. However, if the function is continuous on a closed interval, then we can guarantee the existence of the absolute maximum and minimum:

Extreme Value Theorem: If f is continuous on $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

Corollary: If f is continuous and differentiable on (a, b), and f has exactly one critical point in (a, b) at which a sign change occurs, then f has an absolute maximum or minimum at this point.

An important application of this corollary occurs when we are trying to maximize or minimize a quantity. This type of problem is called an **optimization problem**. For instance, suppose a closed cylindrical can is to hold 1000 cm^3 (1 liter) of liquid, and we want to find the height and radius of the can that requires the least amount of material. Two relevant formulas are



Volume: $V = \pi r^2 h = 1000$ Surface Area: $A = 2\pi r^2 + 2\pi rh$

In order to minimize A, we must write it as a function of one variable, and we can do so by solving the volume equation for h, say, and substituting it into the area equation:

$$A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

All that remains is to find the absolute minimum of the function $A(r) = 2\pi r^2 + \frac{2000}{r}$ ($r > 0$) using the corollary.

Steps for solving an "optimization" problem:

- Draw a sketch labeled with variables.
- Create a list of equations that are relevant to the problem.
- Identify the quantity to be optimized and write a function that models it.
- Consider the domain and find the absolute maximum or minimum of the function.
- Write your answer with units.