Examples 7.4 – The Extreme Value Theorem and Optimization

1. (a) Find the absolute maximum and minimum values of $f(x) = 4x^2 - 12x + 10$ on [1, 3]. State where those values occur.

   (a) Find the absolute maximum and minimum values of $g(x) = x^2 + \frac{2000}{x}$ on (0, +∞), if they exist. State where those values occur.

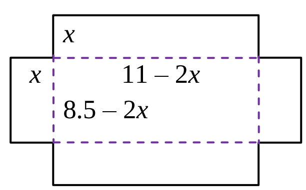
   Solution: (a) By the EVT, we can compare the function values at local extrema and at endpoints. Since $f'(x) = 8x - 12$, the only critical point is x = 1.5. Now, $f(1) = 2$, $f(1.5) = 1$, and $f(3) = 10$, so the absolute maximum is at x = 3 and it is $f(3) = 10$, and the absolute minimum is at x = 1.5 and it is $f(1.5) = 1$.

   (b) Note that g is continuous and differentiable on (0, +∞). After setting $g'(x) = 2x - \frac{2000}{x^2} = 0$, we find that x = 10. A sign test shows that g' is negative to the left of 10 and positive to the right. By the corollary to the EVT, g has an absolute minimum at x = 10, and it is $g(10) = 300$.

2. Suppose a closed cylindrical can is to hold 1000 cm^3 (1 liter) of liquid. Find the height and radius of the can that requires the least amount of material.

   Solution: We must minimize $A(r) = 2\pi r^2 + \frac{2000}{r}$ (r > 0) (see Lesson 7.4). The derivative $A'(r) = 4\pi r - \frac{2000}{r^2}$ is zero when $r = \sqrt[3]{500/\pi}$. A sign test shows that A' is negative to the left of $r = \sqrt[3]{500/\pi}$ and positive to the right. By the corollary to the EVT, A has an absolute minimum value at $r = \sqrt[3]{500/\pi}$. Therefore, a can of radius $r = \sqrt[3]{500/\pi} \approx 5.4$ cm and height $h = 1000/(\pi r^2) \approx 10.8$ cm will require the least amount of material.

3. An open top container is to be made from a piece of 8.5-inch by 11-inch cardboard by cutting out squares of equal size from the four corners and bending up the sides. What length should the squares be to obtain a box with the largest volume?



   Solution: The relevant equation is volume $V = x(11-2x)(8.5-2x) = 4x^3 - 39x^2 + 93.5x$, which is already in terms of one variable x. Note that x cannot be negative, and it also cannot be greater than half the shorter length, or 4.25 inches. We now have a continuous function on a closed interval [0, 4.25]. The derivative of $V(x)$ is $V'(x) = 12x^2 - 78x + 93.5$, which is zero at $x \approx 1.585$ and 4.915. The latter value is outside the domain, so the only critical point is at $x \approx 1.585$. A sign test shows that V' is positive to the left of 1.585 and negative to the right. By the corollary to the EVT, the maximum volume will occur when $x \approx 1.585$ in.