Activity 7.4 – The Extreme Value Theorem and Optimization

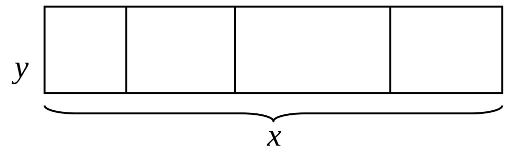
1. Length: $L = 2x + 5y$ (minimize)

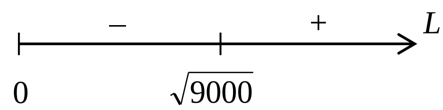
   Area: $A = xy = 3600 \;\rightarrow\; y = \frac{3600}{x}$

   $L(x) = 2x + 5\left(\frac{3600}{x}\right) = 2x + \frac{18000}{x}$

   $L'(x) = 2 - \frac{18000}{x^2} = 0 \;\rightarrow\; x^2 = 9000 \;\rightarrow\; x = \sqrt{9000}$

   Minimum length of fencing is $L\left(\sqrt{9000}\right) \approx 379$ ft.





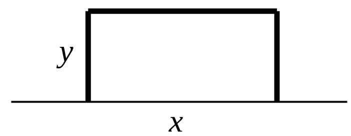
2. Area: $A = xy$ (maximize)

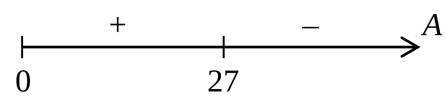
   Cost: $C = 10x + 30y = 540 \;\rightarrow\; y = \frac{540-10x}{30} = \frac{54-x}{3}$

   $A(x) = x\left(\frac{54-x}{3}\right) = 18x - \frac{1}{3}x^2$

   $A'(x) = 18 - \frac{2}{3}x = 0 \;\rightarrow\; x = 27$

   Maximum area is $A(27) = 243$ ft^2.





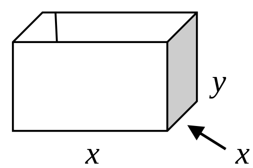
3. Volume: $V = x^2y$ (maximize)

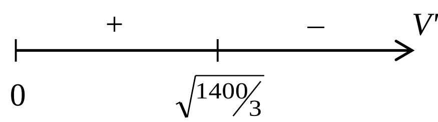
   Area: $A = x^2 + 4xy = 1400 \;\rightarrow\; y = \frac{1400-x^2}{4x}$

   $V(x) = x^2\left(\frac{1400-x^2}{4x}\right) = 350x - \frac{1}{4}x^3$

   $V'(x) = 350 - \frac{3}{4}x^2 = 0 \;\rightarrow\; x^2 = \frac{1400}{3} \;\rightarrow\; x = \sqrt{\frac{1400}{3}}$

   Maximum volume is $V\left(\sqrt{\frac{1400}{3}}\right) \approx 5040$ cm^3.





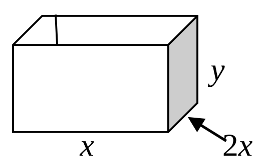
4. Cost: $C = 26x^2 + 30xy$ (minimize)

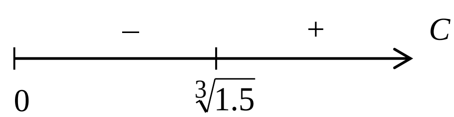
   Volume: $V = 2x^2y = 5.2 \;\rightarrow\; y = \frac{2.6}{x^2}$

   $C(x) = 26x^2 + 30x\left(\frac{2.6}{x^2}\right) = 26x^2 + \frac{78}{x}$

   $C'(x) = 52x - \frac{78}{x^2} = 0 \;\rightarrow\; x^3 = 1.5 \;\rightarrow\; x = \sqrt[3]{1.5}$

   Minimum cost is $C\left(\sqrt[3]{1.5}\right) = \$110.50$.



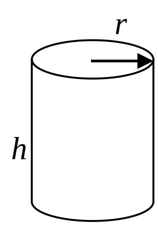


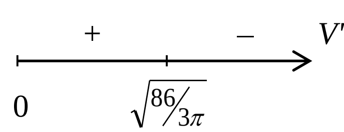
5. Volume: $V = \pi r^2 h$ (maximize)

   Area: $A = \pi r^2 + 2\pi rh = 86 \;\rightarrow\; h = \frac{86-\pi r^2}{2\pi r}$

   $V(r) = \pi r^2\left(\frac{86-\pi r^2}{2\pi r}\right) = 43r - \frac{\pi}{2}r^3$

   $V'(r) = 43 - \frac{3\pi}{2}r^2 = 0 \;\rightarrow\; r^2 = \frac{86}{3\pi} \;\rightarrow\; r = \sqrt{\frac{86}{3\pi}}$





Height is $h = \frac{86 - \pi(86/(3\pi))}{2\pi\sqrt{86/(3\pi)}} \approx 3.02$ in. Maximum liquid is $V\left(\sqrt{\frac{86}{3\pi}}\right) \approx 86.6$ in^3.