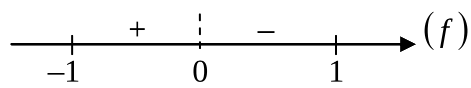
Examples 7.2 – Graph Analysis Using First and Second Derivatives

Let $f(x) = \dfrac{x^2 - 1}{x^3}$. Find each of the following:

(a) Domain, x-intercepts, y-intercept
(b) Vertical asymptotes and nearby behavior
(c) Horizontal asymptotes
(d) Critical points
(e) Intervals of increase/decrease, and local extrema
(f) Intervals of concavity, and inflection
(g) Sketch, or check with graphing device

Solution: (a) The domain of f is $(-\infty,0)\cup(0,+\infty)$, and the x-intercepts are $x = \pm 1$. Since $x = 0$ is not in the domain of f, there is no y-intercept.

(b) The denominator (and not the numerator) is zero at $x = 0$, so the line $x = 0$ is the only vertical asymptote.



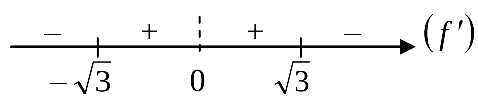
To determine the behavior near $x = 0$, we plot the x-intercepts and vertical asymptotes, and test f for sign on either side of and between these points. Since $f(x) > 0$ for all x in (–1, 0), and $f(x) < 0$ for all x in (0, 1) it follows that $\lim_{x\to 0^-} \frac{x^2-1}{x^3} = +\infty$ and $\lim_{x\to 0^+} \frac{x^2-1}{x^3} = -\infty$.

(c) Since $\lim_{x\to -\infty} \frac{x^2-1}{x^3} = 0 = \lim_{x\to +\infty} \frac{x^2-1}{x^3}$, the only horizontal asymptote is $y = 0$.

(d) The critical points are found by analyzing the derivative, which is $f'(x) = \frac{3-x^2}{x^4}$ (check!).

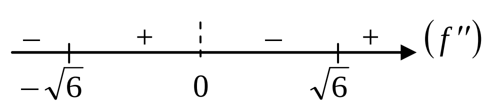
Note that $f'(x)$ is undefined at $x = 0$, but this is not a critical point since it is not in the domain of f. Therefore, the critical points are $x = \pm\sqrt{3}$, the zeros of the numerator.

(e) To find the intervals of increase and decrease, we plot the critical points and vertical asymptotes, and test for sign on either side of and between these points.



The signs show that f is increasing on $\left(-\sqrt{3},0\right)\cup\left(0,\sqrt{3}\right)$ and decreasing on $\left(-\infty,-\sqrt{3}\right)\cup\left(\sqrt{3},+\infty\right)$. By the first derivative test, f has a local minimum at $x = -\sqrt{3}$ and it is $f\left(-\sqrt{3}\right) \approx -0.385$. Similarly, f has a local maximum at $x = \sqrt{3}$ and it is $f\left(\sqrt{3}\right) \approx 0.385$.

(f) The second derivative of f is $f''(x) = \frac{2x^2-12}{x^5}$ (check!).



To find intervals of concavity, we find the zeros and vertical asymptotes of f'', and test for sign on either side of and between these points. We see that f is concave up on $\left(-\sqrt{6},0\right)\cup\left(\sqrt{6},+\infty\right)$ and concave down on $\left(-\infty,-\sqrt{6}\right)\cup\left(0,\sqrt{6}\right)$. From the sign chart, we find that the inflection points are at $x = \pm\sqrt{6}$ and are (approximately) $\left(-\sqrt{6},-0.340\right)$ and $\left(\sqrt{6},0.340\right)$.