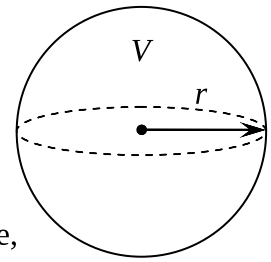
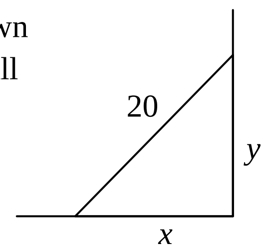
Examples 7.1 – Related Rates

1. The radius of a spherical balloon is decreasing at a constant rate of 1.5 cm/min. How fast is air being released when the radius is 20 cm?



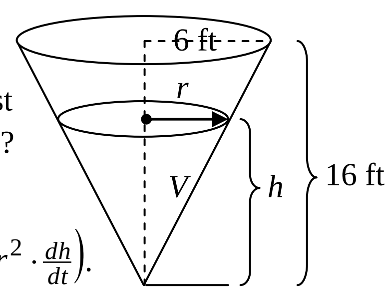
Solution: An equation that relates V and r is the formula for the volume of the sphere, $V = \frac{4}{3}\pi r^3$. We are given that $\frac{dr}{dt} = -1.5$ cm/min, and we want $\frac{dV}{dt}$ when r = 20 cm. By implicit differentiation with respect to time, we have $\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$, and by substitution, $\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3(20)^2 \cdot (-1.5) \approx -7540$. That is, when the radius is 20 cm, air is being released at a rate of about 7540 cm^3/min.

2. As a 20-ft ladder leans against a wall, the top of the ladder is slipping down the wall at a rate of 2 ft/s. How fast is the foot moving away from the wall when the top is 10 ft above the ground?



Solution: An equation that relates x and y is the Pythagorean Theorem, $x^2 + y^2 = 400$. We are given that $\frac{dy}{dt} = -2$ ft/s, and we want $\frac{dx}{dt}$ when y = 10 ft. By implicit differentiation with respect to time t, we have $2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$. Note that we are not given x when y = 10, but we can find it from the related variables equation. Since $x^2 + 10^2 = 400$, it follows that $x = 10\sqrt{3}$. By substitution, we have $2(10\sqrt{3}) \cdot \frac{dx}{dt} + 2(10) \cdot (-2) = 0$, and so $\frac{dx}{dt} = \frac{2}{\sqrt{3}} \approx 1.15$. Therefore, when the top of the ladder is 10 ft above the ground, the foot is moving away from the wall at a rate of about 1.15 ft/s.

3. Suppose water is draining from a conical tank. The radius of the top of the tank is 6 ft, and the height of the tank is 16 ft. If the water flows out of the tank at a rate of 10 ft^3/min, how fast is the height of the water decreasing when the water is 8 ft deep?



Solution: From Lesson 7.1, the related variables equation is $V = \frac{\pi}{3} r^2 h$, and the related rates equation is $\frac{dV}{dt} = \frac{\pi}{3}\left(2r \cdot \frac{dr}{dt} \cdot h + r^2 \cdot \frac{dh}{dt}\right)$. We are given that $\frac{dV}{dt} = -10\,\text{ft}^3$/min, and we want $\frac{dh}{dt}$ when h = 8 ft. Note that we do not know r and $\frac{dr}{dt}$ when h = 8 ft. To find these quantities, we need another equation that relates r and h. A logical choice is the one that is obtained by similar triangles, namely $\frac{r}{6} = \frac{h}{16}$, or $r = \frac{3}{8}h$. When h = 8, it follows that r = 3, and by differentiation $\frac{dr}{dt} = \frac{3}{8} \cdot \frac{dh}{dt}$. By substitution, we have $-10 = \frac{\pi}{3}\left(2(3) \cdot \left(\frac{3}{8} \cdot \frac{dh}{dt}\right) \cdot (8) + (3)^2 \cdot \frac{dh}{dt}\right)$, and so $\frac{dh}{dt} = -0.35$. Therefore, when the water is 8 ft deep, the height of the water is decreasing by about 0.35 ft/min.