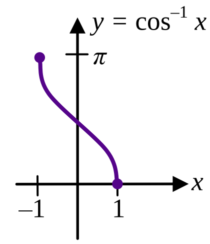
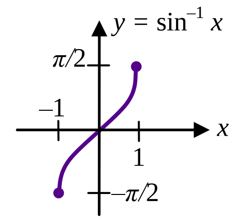
Lesson 6.4 – Inverse Trigonometric Functions

The graphs of cosine, sine, and tangent all fail the horizontal line test. However, we can restrict their domains—that is, only consider a piece of each graph—to create useful inverse functions.



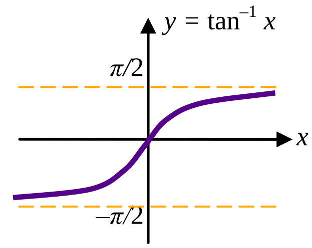
Inverse cosine function: $y = \cos^{-1} x = \arccos x$

- Think of $\cos^{-1} x$ as the angle whose cosine is *x*.
- Restrict the domain of cosine to $[0, \pi]$, and reflect about $y = x$.
- The domain is $[-1, 1]$, and the range is $[0, \pi]$.



Inverse sine function: $y = \sin^{-1} x = \arcsin x$

- Think of $\sin^{-1} x$ as the angle whose sine is *x*.
- Restrict the domain of sine to $[-\pi/2, \pi/2]$, and reflect about $y = x$.
- The domain is $[-1, 1]$, and the range is $[-\pi/2, \pi/2]$.



Inverse tangent function: $y = \tan^{-1} x = \arctan x$

- Think of $\tan^{-1} x$ as the angle whose tangent is *x*.
- Restrict the domain of tangent to $(-\pi/2, \pi/2)$, and reflect about $y = x$.
- The domain is $(-\infty, +\infty)$, and the range is $(-\pi/2, \pi/2)$.
- The vertical asymptotes of $\tan x$ at $x = \pm\pi/2$ become horizontal asymptotes of $\tan^{-1} x$. Therefore, $\lim_{x\to+\infty} \tan^{-1} x = \frac{\pi}{2}$ and $\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$.

Derivatives of the inverse trigonometric functions and corresponding antiderivatives:

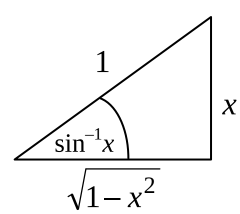
$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}} \qquad \int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

Let us verify the derivative formula for $\sin^{-1} x$ and derive the others in Activity 6.4. By the derivative-of-an-inverse formula,

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\cos(\sin^{-1} x)}$$



Since $\sin^{-1} x$ is the angle whose sine is *x*, we can construct a right triangle that satisfies this condition, and then find the cosine of that angle. Hence, $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$, and we are done.