Lesson 5.6 – Definite Integrals of Exponentials and Logarithms

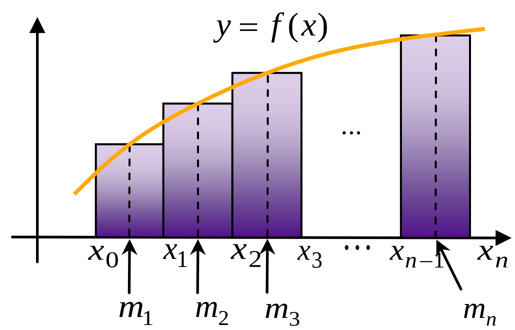
If f is a constant or linear function, then the FTC says that the net area between the graph of f and the x-axis on the interval $[a, b]$ is the same as the change in any antiderivative F on $[a, b]$:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

The same is true for exponentials and logarithms, and we will verify this in Chapter 8. This will take some work since a region bounded by an exponential or logarithm does not necessarily have a geometrical formula for its area. For now, the idea is to cover the region systematically with rectangles so that the area can be approximated to any degree of accuracy. We investigated this idea in Activity 2.6 using left- and right-hand approximations, but a third option is to use the midpoint of each subinterval. Here are the formal steps for the **midpoint approximation** of the area bounded by a polynomial, rational, exponential, or logarithmic function f that is continuous on the interval $[a, b]$:

STEP 1: Let $x_0 = a$ and $x_n = b$ and divide $[x_0, x_n]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$:

$$[x_0, x_1]\ ,\ [x_1, x_2]\ ,\ \ldots\ ,\ [x_{k-1}, x_k]\ ,\ \ldots\ ,\ [x_{n-1}, x_n]$$



STEP 2: For each $k \geq 1$, choose the midpoint in the kth subinterval $[x_{k-1}, x_k]$:

First midpoint: $\quad m_1 = x_0 + \Delta x - \frac{1}{2}\Delta x = x_0 + \left(1 - \frac{1}{2}\right)\Delta x$

Second midpoint: $\quad m_2 = x_0 + 2\Delta x - \frac{1}{2}\Delta x = x_0 + \left(2 - \frac{1}{2}\right)\Delta x$

$\vdots$

kth midpoint: $\quad m_k = x_0 + \left(k - \frac{1}{2}\right)\Delta x$

STEP 3: For each $k \geq 1$, construct the kth rectangle of height $f(m_k)$ and width Δx:

Area of kth rectangle = $f(m_k)\Delta x$

STEP 4: Add up the areas of the n rectangles to get an approximation A_n of the net signed area:

$$\int_a^b f(x)dx \approx A_n = f(m_1)\Delta x + \cdots + f(m_n)\Delta x = \big(f(m_1) + \cdots + f(m_n)\big)\Delta x$$

We could write the sum more compactly using sigma notation (to be discussed in Lesson 8.1) as

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} f(m_k)\Delta x \qquad \text{(a \textbf{Riemann sum})}$$