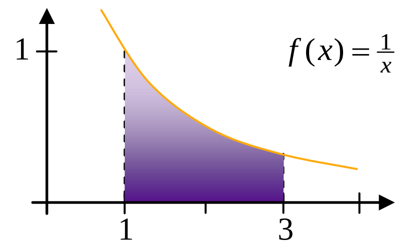
Examples 5.6 – Definite Integrals of Exponentials and Logarithms

1. (a) Use a midpoint approximation and n = 8 subintervals to approximate the net area bounded by the graph of $f(x) = \frac{1}{x}$ and the x-axis on [1, 3].



(b) Use the FTC to find the exact value of $\int_1^3 \frac{1}{x}\,dx$.

Solution:

(a) **STEP 1**: Divide [1, 3] into n = 8 subintervals of equal width, $\Delta x = \frac{3-1}{8} = \frac{2}{8} = \frac{1}{4}$.

STEP 2: For each $k \geq 1$, the midpoint of the kth subinterval $[x_{k-1}, x_k]$ is

$$m_k = 1 + \left(k - \tfrac{1}{2}\right)\tfrac{1}{4} = 1 + \tfrac{k}{4} - \tfrac{1}{8} = \tfrac{7+2k}{8}.$$

STEP 3: The area of the kth rectangle having height $f(m_k) = \frac{1}{m_k} = \frac{8}{7+2k}$ and width

$$\Delta x = \tfrac{1}{4} \text{ is } f(m_k)\cdot\Delta x = \tfrac{8}{7+2k}\cdot\tfrac{1}{4}.$$

STEP 4: Therefore, $\int_1^3 \frac{1}{x}\,dx \approx \left(\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15} + \frac{8}{17} + \frac{8}{19} + \frac{8}{21} + \frac{8}{23}\right)\cdot\frac{1}{4} \approx 1.096325$.

(a) By the FTC, the exact value is $\int_1^3 \frac{1}{x}\,dx = \left(\ln|x|\right)\Big|_1^3 = \ln|3| - \ln|1| = \ln 3 \approx 1.098612$.

Additional note: The table below shows left-hand, midpoint, and right-hand approximations of $\int_1^3 \frac{1}{x}\,dx$ for n = 4, 8, 100, and 1000 subintervals. Note that in this example the midpoint approximation requires "only" 100 rectangles for four decimal place accuracy.

n	Left-hand	Midpoint	Right-hand	Exact
4	1.283333	1.089755	0.950000	1.098612…
8	1.186544	1.096325	1.019877	
100	1.105309	1.098597	1.091975	
1000	1.099279	1.098612	1.097945	

2. An object in rectilinear motion has velocity given by $v(t) = e^t$ cm/min. Find the displacement and the total distance traveled during the first four minutes.

Solution: Since $v(t)$ is positive on the given interval, the displacement and total distance traveled are both equal to

$$\int_0^4 e^t\,dt = e^t\Big|_0^4 = e^4 - e^0 = e^4 - 1 \approx 53.6\text{ cm}$$