# Lesson 5.5 – Derivatives and Antiderivatives of Exponentials and Logarithms

Instead of using the definition of the derivative to compute the derivative of the natural logarithm, we could use the fact that it is the inverse of the natural exponential and apply the derivative-of-an-inverse rule. In the general case, we could change the base to base *e* first. Here are the details:

**Derivative of the natural logarithm:** Let $f(x) = e^x$ so that $f'(x) = e^x$ and $f^{-1}(x) = \ln x$. Then

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

**Derivative of the logarithm base *b*:** After performing a change of base, we have

$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b}\cdot\frac{d}{dx}(\ln x) = \frac{1}{(\ln b)x}$$

**Derivative of the exponential base *b*:** After performing a change of base, the chain rule gives

$$\frac{d}{dx}\left(b^x\right) = \frac{d}{dx}\left(e^{(\ln b)x}\right) = e^{(\ln b)x}\cdot\ln b = (\ln b)\cdot b^x$$

By dividing both sides of the last equation by ln *b*, we can derive the corresponding integral. Here is a summary of formulas for exponential functions:

**Derivatives and antiderivatives of exponential functions:**

$$\frac{d}{dx}(e^x) = e^x \qquad \int e^x dx = e^x + C$$

$$\frac{d}{dx}(b^x) = (\ln(b))\cdot b^x \qquad \int b^x dx = \frac{b^x}{\ln(b)} + C$$

**Antiderivative of $x^{-1}$:** Since $\frac{d}{dx}(\ln x) = \frac{1}{x}$ $(x > 0)$ and $\frac{d}{dx}(\ln(-x)) = \frac{1}{-x}(-1) = \frac{1}{x}$ $(x < 0)$, we have

$$\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C$$

This last formula is the missing piece of the power rule for integration:

**Complete power rule for integration:** $\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + C, & \text{if } n \neq -1 \\ \ln|x| + C, & \text{if } n = 1 \end{cases}$