Examples 5.4 – Logarithmic Functions

1. Use the properties of logarithms to expand the expression $\ln\left(\frac{x^3\sqrt{4x-11}}{(1+x^2)^5}\right)$.

   Solution: By the algebraic properties of logarithms,

$$\begin{aligned}\ln\left(\frac{x^3\sqrt{4x-11}}{(1+x^2)^5}\right) &= \ln\left(x^3\sqrt{4x-11}\right) - \ln\left(\left(1+x^2\right)^5\right)\\ &= \ln\left(x^3\right) + \ln\left(\sqrt{4x-11}\right) - \ln\left(\left(1+x^2\right)^5\right)\\ &= 3\ln x + \tfrac{1}{2}\ln(4x-11) - 5\ln\left(1+x^2\right)\end{aligned}$$

2. Convert 5^x and $\log_3 x$ to base e.

   Solution: We have $5^x = e^{(\ln 5)x}$, and $\log_3 x = \frac{\ln x}{\ln 3} = \left(\frac{1}{\ln 3}\right)\cdot \ln x$.

3. Use inverse properties to solve the equation for x.

   (a) $e^{x^2-4} = 2$ (b) $\log_{10}(3x+1) = -1$

   Solution: (a) If $e^{x^2-4} = 2$, then $\ln 2 = \ln\left(e^{x^2-4}\right) = x^2 - 4$, thus $x = \pm\sqrt{\ln 2 + 4}$.

   (b) If $\log_{10}(3x+1) = -1$, then $10^{-1} = 10^{\log_{10}(3x+1)} = 3x+1$, and so $x = -\frac{3}{10}$.

4. Suppose you invest \$1000 in an account that earns 6% annual interest compounded monthly. Write a discrete growth model for the amount in the account after the t-th year. Then determine the **doubling time**, or how many months before the account doubles.

   Solution: The model is $A(t) = 1000\left(1+\frac{0.06}{12}\right)^{12t} = 1000(1.005)^{12t}$ dollars after year t. To find the number of months before the account will have \$2000, we set $1000(1.005)^{12t} = 2000$ and solve for t. We have,

$$\begin{aligned}1000(1.005)^{12t} &= 2000\\ (1.005)^{12t} &= 2\\ \ln\left((1.005)^{12t}\right) &= \ln(2)\\ 12t\cdot\ln(1.005) &= \ln(2)\end{aligned}$$

   Therefore,

$$t = \frac{\ln(2)}{12\ln(1.005)} \approx 11.58 \text{ years} \approx 139 \text{ months}$$