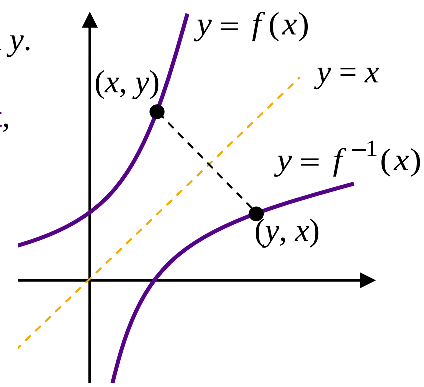
Lesson 5.3 – Implicit Differentiation and Inverse Functions

A function y is defined **explicitly** in terms of x if it is written in the form $y = f(x)$. In other words, the only y in the equation appears on the left. These are the types of functions that we have studied so far. Finding the derivative of an explicitly defined function is simply a matter of applying the appropriate differentiation rules and formulas to the right-hand side.

A function y is defined **implicitly** if its relationship to the variable x is given by an equation that is not solved explicitly for y. A simple example is the equation of a **circle** centered at the origin of radius r, namely $x^2 + y^2 = r^2$. In this case, we can find the derivative of y by first solving for y in terms of x, but in general, this is not always reasonable or even possible. The good news is that we can find a formula for the derivative of y without ever knowing its explicit form.

Implicit differentiation: Suppose we are given an implicit equation containing x and y such that y is a function of x, and we want $\frac{dy}{dx} = y'$.

> **Step 1**: Differentiate both sides with respect to x, and use the chain rule on each expression involving y. That is, use $\frac{d}{dx}(g(y)) = g'(y) \cdot y'$. Thus y' will appear in one or more terms.
>
> **Step 2**: Solve for y'. The result will usually contain both x and y.



Inverse functions: If the graph of f passes the **horizontal line test**, then the graph obtained by reversing the inputs and outputs of f passes the **vertical line test**. This graph represents a new function called the **inverse** of f, denoted by f^{-1}, and we say that f is an **invertible function**. The graph of f^{-1} is the graph of f reflected about the line $y = x$. Sometimes the domain must be restricted so that only part of the function is invertible. For instance, $\sqrt{x}$ is the inverse of x^2 if $x \geq 0$. If an invertible function f is given by an equation $y = f(x)$, then a formula for f^{-1} can be found by solving for x to get $x = f^{-1}(y)$. To graph the inverse in the Cartesian plane, we interchange x and y and write $y = f^{-1}(x)$. Inverses "undo" each other under function composition:

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x \qquad \text{and} \qquad (f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$$

Derivative of an inverse function: If $y = f^{-1}(x)$, then $x = f(y)$. By implicit differentiation and the chain rule, $1 = \frac{d}{dx}(f(y)) = f'(y) \cdot y' = f'(f^{-1}(x)) \cdot y'$, hence

$$y' = \frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$