Examples 5.3 – Implicit Differentiation and Inverse Functions

1. Determine the following derivatives. Assume that y is a function of x.

   (a) $\frac{d}{dx}\left(x^2\right)$ (b) $\frac{d}{dx}\left(y^2\right)$ (c) $\frac{d}{dx}\left(x^2\sqrt{y}\right)$

   Solution: (a) $\frac{d}{dx}\left(x^2\right) = 2x$

   (b) $\frac{d}{dx}\left(y^2\right) = 2y \cdot y' = 2y \cdot \frac{dy}{dx}$

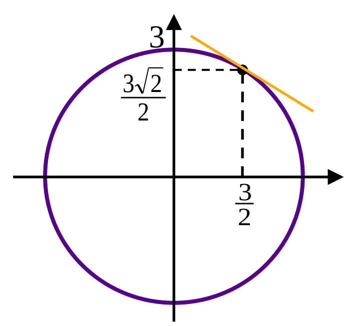
   (c) $\frac{d}{dx}\left(x^2\sqrt{y}\right) = \frac{d}{dx}\left(x^2\right) \cdot \sqrt{y} + x^2 \cdot \frac{d}{dx}\left(\sqrt{y}\right) = 2x \cdot \sqrt{y} + x^2 \cdot \frac{1}{2\sqrt{y}} \cdot y'$

2. As the volume V of a sphere changes over time t, its radius r also changes. Given that the volume of a sphere is $V = \frac{4}{3}\pi r^3$, find a formula for the rate of change of the radius with respect to time.

   Solution: By implicit differentiation, $\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$ and $\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$. Therefore, $\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt}$.

3. Find the equation of the tangent line to the circle $x^2 + y^2 = 9$ at $x_0 = \frac{3}{2}$ in the first quadrant.

   Solution: By implicit differentiation, $2x + 2yy' = 0$. That is, $2yy' = -2x$, and so $y' = -\frac{x}{y}$. The y-coordinates of the points on the circle at $x_0 = \frac{3}{2}$ are $y = \pm\frac{3\sqrt{2}}{2}$. In the first quadrant, $y_0 = \frac{3\sqrt{2}}{2}$, hence the slope is $y'\big|_{(x_0, y_0)} = -\frac{\frac{3}{2}}{\frac{3\sqrt{2}}{2}} = -\frac{1}{\sqrt{2}}$. In point-slope form, the equation of the tangent line is $y - \frac{3\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}\left(x - \frac{3}{2}\right)$.



4. Find the inverse function for $y = f(x) = 3x - 7$.

   Solution: Solving for x yields $x = \frac{1}{3}(y + 7)$, hence $f^{-1}(x) = \frac{1}{3}(x + 7)$. Note that f and f^{-1} "undo" each other under function composition. That is,

$$(f \circ f^{-1})(x) = 3\left(\tfrac{1}{3}(x + 7)\right) - 7 = x \quad \text{and} \quad (f^{-1} \circ f)(x) = \tfrac{1}{3}\left((3x - 7) + 7\right) = x$$