Lesson 5.2 – Derivative and Antiderivative of e^x

Although limits have played a vital role for us, we have intentionally avoided a direct discussion of the topic. Our intuition, along with numerical evidence, has helped us to gain a fairly good understanding of how limits work. Students who take courses in "advanced calculus" or "real analysis" will study the precise definition of the limit and prove why our intuition is correct. As we encounter more complicated functions, we will need a more focused list of facts about limits. As before, use your intuition to help justify their validity.

Limit laws: Assume the existence of all limits below.

1. The limit of a **constant** is itself.
2. The limit of a **sum/difference/product** is the sum/difference/product of the limits, resp.
3. The limit of a **quotient** is the quotient of the limits if the limit of the denominator is not zero.
4. The limit of an ***n*th root** is the root of the limit (the limit must be nonnegative if n is even).

Derivative and antiderivative of $y = e^x$: By the definition of derivative and the fact that $\lim_{\Delta x\to 0} \frac{e^{\Delta x}-1}{\Delta x} = 1$ (Activity 5.1), we have

$$\frac{d}{dx}\left(e^x\right) = \lim_{\Delta x\to 0} \frac{e^{x+\Delta x}-e^x}{\Delta x} = \lim_{\Delta x\to 0} \frac{e^x(e^{\Delta x}-1)}{\Delta x} = \lim_{\Delta x\to 0} e^x \cdot \lim_{\Delta x\to 0} \frac{e^{\Delta x}-1}{\Delta x} = e^x \cdot 1 = e^x$$

In other words, the slope and the height of the graph of $y = e^x$ at x are numerically equal to each other! (The units are different.) Conversely, since $\frac{d}{dx}\left(e^x\right) = e^x$, it follows that $\int e^x dx = e^x + C$.

The function $y = e^{kx}$ arises frequently in applications, so we pay a little more attention to it here.

Derivative and antiderivative of $y = e^{kx}$: By the chain rule, $\frac{d}{dx}(e^{kx}) = k \cdot e^{kx}$, so we "pick up" a factor of k. Also by the chain rule, $\frac{d}{dx}\left(\frac{1}{k}\cdot e^{kx}\right) = e^{kx}$, and it follows that $\int e^{kx} dx = \frac{1}{k}\cdot e^{kx} + C$. For integration, we "pick up" a factor of $1/k$.

Another way to justify the antiderivative of the composition $y = e^{kx}$ is to think of the "inside" function kx as a new variable $u = kx$, and then integrate $\int e^u dx$. The problem, however, is that we cannot integrate a function of u with respect to x. We need du in the integrand. The remedy is to rewrite dx in terms of u and substitute it into the original integral. By differentiation, we have

$$u = kx \quad \to \quad \frac{du}{dx} = k \quad \to \quad dx = \frac{1}{k}\cdot du$$

Therefore, $\int e^{kx} dx = \int e^u \cdot \frac{1}{k} \cdot du = \frac{1}{k}\int e^u du = \frac{1}{k} e^u + C = \frac{1}{k} e^{kx} + C$. This process is called **integration by substitution**, or ***u*-substitution**, and we will study it formally in Lesson 8.6.