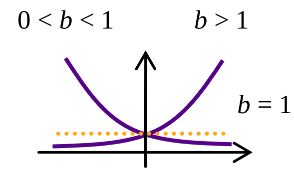
Lesson 5.1 – Exponential Growth and Decay

> An **exponential function with base *b*** is a function of the form $y = b^x$, where $b > 0$ is a fixed real constant.

Do not confuse exponential functions with power functions:

$$\text{Power function: } y = x^b = (\text{variable base})^{(\text{constant power})}$$

$$\text{Exponential function: } y = b^x = (\text{constant base})^{(\text{variable power})}$$



Domain: The set of all real numbers.

Range: (The **range** of a function is the set of all possible outputs.) For an exponential function, the range is all positive real numbers, or $(0, \infty)$. Therefore, the graph lies above the x-axis.

***y*-intercept:** Set $x = 0$ to get $y = b^0 = 1$ for all $b > 0$. Therefore, the y-intercept is (0, 1).

***x*-intercept:** Since $b^x = 0$ has no solution, an exponential function has no x-intercepts

Graph: Everywhere continuous for any $b > 0$. If $b = 1$, then $y = 1^x = 1$ for all x, so in this case the exponential function is the constant $y = 1$. When $b > 1$, the graph is positive, increasing, and concave up. When $0 < b < 1$, the graph is positive, decreasing, and concave up. The x-axis is a horizontal asymptote.

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Discrete growth/decay model: Suppose an initial amount A is growing or decaying at a constant rate of R% per unit time. Let r denote the decimal form of R%, and let $r < 0$ denote decay. Then for consecutive time units, the amount present at the later time is $(100 \pm R)$% of the amount present at the earlier time. As a decimal, this is equivalent to multiplication by $(1 + r)$, and so the amount present at time t is given by $y(t) = A(1+r)^t$. This model assumes that the growth/decay rate is compounded once per unit time. If the rate is compounded n times per unit time, then the rate r is divided by n and the exponent is multiplied by n. In this case, $y(t) = A\left(1+\frac{r}{n}\right)^{nt}$.

Continuous growth/decay model: As the number n of compounds per unit time tends to infinity, we get the continuous model,

$$y(t) = \lim_{n\to+\infty} A\left(1+\tfrac{r}{n}\right)^{nt} = A\cdot\left(\lim_{n\to+\infty} A\left(1+\tfrac{r}{n}\right)^{n}\right)^{t}$$

In Activity 5.1, we will numerically verify that $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e \approx 2.718282$ (**Euler's number**), and it turns out that $\lim_{n\to+\infty}\left(1+\frac{r}{n}\right)^n = e^r$. The continuous model becomes $y(t) = A\left(e^r\right)^t = Ae^{rt}$.

The "natural" base: If e is Euler's number, then the function $y = e^x = \exp(x)$ is the exponential function base e. It is called the **natural exponential function**.