Activity 5.1[†] – Exponential Growth and Decay

FOR DISCUSSION: *What is the difference between exponential and power functions?*
How are exponential and linear models similar? How are they different?

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1. A patient is injected with a 500 mg dose of a medication. After 2 hours, 245 mg remain in the bloodstream. Let $M(t)$ be the amount of medication in the bloodstream (in mg) t hours after the injection. Notice that you are given two points (0, 500) and (2, 245).

   (a) Write M as a linear function of the form $M(t) = at + b$. (Find a and b.)

   (b) Write M as a discrete exponential function of the form $M(t) = A(1 - r)^t$. (Find A and $1–r$.)

2. A certain radioactive material decays in such a way that the mass remaining (in grams) after t years is given by the function $m(t) = 90e^{-0.03t}$.

   (a) Determine the initial mass of the material.

   (b) Compute the mass remaining after 40 years.

   (c) Determine the decay rate for this material (as a nonnegative percentage).

[†] This activity is referenced in Lessons 5.1 and 5.2

3. World poultry production was 77.2 million tons in the year 2004 and increasing at a continuous rate of 1.6% per year. Write an exponential model $P(t)$ for world poultry production in million tons, where t is years since 2004.

4. Suppose you invest A = \$1.00 at r = 100% interest compounded n times per year. The discrete model for this situation is

$$y(t) = A \cdot \left(1 + \tfrac{r}{n}\right)^{n\,t} = (1.00) \cdot \left(1 + \tfrac{1.00}{n}\right)^{n\,t} = \left(1 + \tfrac{1}{n}\right)^{n\,t}$$

After t = 1 year, the amount of money in your account is

$$y(1) = \left(1 + \tfrac{1}{n}\right)^{n}$$

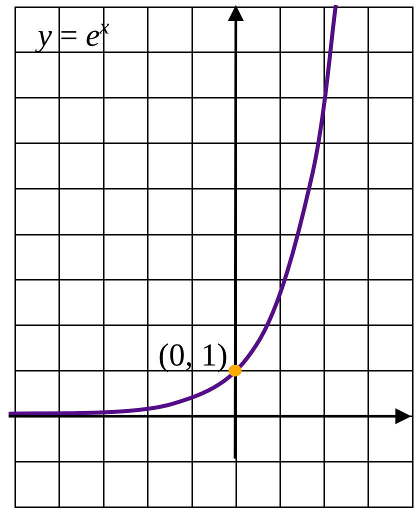
Compounding more and more frequently will, of course, yield more and more money, but the amount you can accumulate over the course of a year has a limit, called <u>Euler's number</u>.

Use your calculator to fill in the table for the given number n of compounds per year. Round your answers to 8 decimal places.

n compounding per yr	$\left(1+\tfrac{1}{n}\right)^n$ dollars after 1 yr
1 (yearly)	$\left(1+\tfrac{1}{1}\right)^1$ = \$2.00000000
12 (monthly)	$\left(1+\tfrac{1}{12}\right)^{12}$ =
365 (daily)	
525,600 (each minute)	
↓	↓
+∞	\$2.718281828459…

Euler's number is defined as $e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = 2.7182818....$

5. In the next lesson, we will need the slope of $f(x) = e^x$ at 0. We can approximate it using two methods.



METHOD 1: Sketch a tangent line to $f(x) = e^x$ at $x = 0$, and approximate its slope using rise over run.

$$f'(0) = \frac{d}{dx}\left(e^x\right)\Big|_{x=0} = \_\_\_\_\_\_\_\_\_\_$$

METHOD 2: By the definition of the derivative at $x = 0$,

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}.$$

Use your calculator to fill in the table for the given values of Δx. Round your answers to 4 decimal places.

Δx	–0.1	–0.01	–0.001	→	0	←	0.001	0.01	0.1
$\frac{e^{\Delta x} - 1}{\Delta x}$				→		←			

Now use the table to deduce the slope of $f(x) = e^x$ at $x = 0$:

$$f'(0) = \frac{d}{dx}\left(e^x\right)\Big|_{x=0} = \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = \_\_\_\_\_\_\_\_\_\_$$

6. (**OPTIONAL**) In Part 1, we saw that $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$. Now let us consider $\lim_{n \to +\infty} \left(1 + \frac{r}{n}\right)^n$ for a fixed $r > 0$. Let $n = mr$. Then as $n \to +\infty$, so does m. Substitute $n = mr$ into $\lim_{n \to +\infty} \left(1 + \frac{r}{n}\right)^n$ and simplify the result to obtain a limit as $m \to +\infty$. Deduce that $\lim_{n \to +\infty} \left(1 + \frac{r}{n}\right)^n = e^r$.