# Lesson 4.3 – Continuity and L'Hôpital's Rule

In Activity 4.2, we deduced that a function $f$ is **continuous at a point $x = a$** if

1. $f(a)$ is defined  ($a$ is in the domain of $f$);
2. $\lim_{x \to a} f(x)$ exists (the limits from the left and right as $x$ approaches $a$ are equal);
3. $\lim_{x \to a} f(x) = f(a)$ (the limit as $x$ approaches $a$ and the output at $a$ are equal).

Sometimes the third condition is taken as the definition of continuity at $x = a$ since the other two conditions could be reasoned from it.

A function $f$ is **continuous on an interval $(a, b)$** if $f$ is continuous at each number in $(a, b)$. If $f$ is continuous on the entire real line, then we say that $f$ is **continuous everywhere**.

Many times we are interested in the behavior of a function at a point $x = a$, say. If the function happens to be continuous at $a$, then simply plugging in $a$ for $x$ will do the trick. Most of the time, however, we are interested in the behavior of a function near a discontinuity, and this requires that we sneak up on $a$ using limits. Although plugging in $a$ for $x$ at a discontinuity of a rational function might give us some information about the limit, it may also yield an **indeterminate form** such as 0/0 or $\pm\infty/\pm\infty$. In other words, the numerator and denominator are both approaching zero or becoming infinite at the same time, but the limit is ultimately determined by who wins the race. According to **L'Hôpital's Rule**, we can use derivatives to evaluate the indeterminate forms 0/0 or $\pm\infty/\pm\infty$. We verify this claim in the case $\lim_{x \to a} \frac{N(x)}{D(x)} = \frac{0}{0}$ where $a$ is finite. Since $N$ and $D$ are polynomials, we can guarantee that each has a tangent line at $x = a$. Rather than follow the functions toward $x = a$, we follow the tangent lines. Note that in this case, $N(a) = D(a) = 0$, so

$$\lim_{x \to a} \frac{N(x)}{D(x)} = \lim_{x \to a} \frac{N(a) + N'(a)(x-a)}{D(a) + D'(a)(x-a)} = \lim_{x \to a} \frac{N'(a)}{D'(a)} = \lim_{x \to a} \frac{N'(x)}{D'(x)}$$

At this point, we will give the formal statement of L'Hôpital's Rule so that we may use it whenever necessary throughout the remainder of the course.

**L'Hôpital's Rule (formal statement):** Suppose that $f$ and $g$ have derivatives on an open interval containing $a$ except possibly at $a$, and that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty} \quad \text{(the rule may fail otherwise!)}$$

If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists or is $\pm\infty$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$. The rule also holds for limits from the right, limits from the left, and limits at infinity or negative infinity.