Examples 4.3 – Continuity and L'Hôpital's Rule

1. (a) Give an example of a function that is continuous but not differentiable.
   (b) Give an example of a function that is differentiable but not continuous.

   Solution: (a) An example of a continuous and non-differentiable function is $f(x) = |x|$. On one hand, we have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0 = f(0)$, which shows that f is continuous at $x = 0$. On the other hand, $-1 = \lim_{x\to 0^-} f'(x) \neq \lim_{x\to 0^+} f'(x) = +1$ which shows that f is not differentiable at $x = 0$.

   (b) There is no such function, since a differentiable function must be continuous! Recall, a function f is differentiable at $x = a$ if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. Since the denominator is approaching zero, the numerator must be approaching zero as well (otherwise the limit would not exist). That is, $\lim_{x\to a}\left(f(x)-f(a)\right) = 0$. But if this is the case, then $\lim_{x\to a} f(x) = f(a)$, which shows that f is continuous at a.

2. Find constants c and d that make the piecewise function f continuous everywhere.

$$f(x) = \begin{cases} 2-x \ , & x < 1 \\ cx^2 + d \ , & 1 \le x < 2 \\ \frac{1}{2}x + 2 \ , & x \ge 2 \end{cases}$$

   Solution: We need the breakpoints to match up. At $x = 1$, we must have $2 - (1) = c(1)^2 + d$, hence, $d = 1 - c$. At $x = 2$, we must have $c(2)^2 + d = \frac{1}{2}(2) + 2$, hence, $d = 3 - 4c$. It follows that $1 - c = 3 - 4c$, so $c = \frac{2}{3}$ and $d = \frac{1}{3}$.

3. Use L'Hôpital's Rule to find $\lim_{x\to 2} \frac{x^2-x-2}{3x^2-9x+6}$ and $\lim_{x\to +\infty} \frac{4x^2-2x+7}{3x^2+9x-1}$.

   Solution: We can use L'Hôpital's Rule for the first limit since direct substitution yields the indeterminate form 0/0. Therefore,

$$\lim_{x\to 2} \frac{x^2-x-2}{3x^2-9x+6} \overset{LR}{=} \lim_{x\to 2} \frac{2x-1}{6x-9} = 1.$$

   For the second limit, both the numerator and denominator grow without bound as x approaches positive infinity, so L'Hôpital's Rule applies here as well. Note that we can repeatedly apply the rule as long as the new limit is the correct indeterminate form:

$$\lim_{x\to +\infty} \frac{4x^2-2x+7}{3x^2+9x-1} \overset{LR}{=} \lim_{x\to +\infty} \frac{8x-2}{6x+9} \overset{LR}{=} \lim_{x\to +\infty} \frac{8}{6} = \frac{4}{3}$$