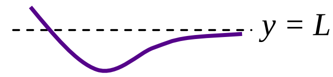
Lesson 4.2 – Horizontal and Vertical Asymptotes

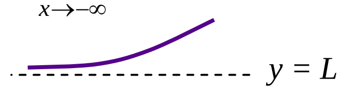
In Lesson 3.2, we learned that **end behavior** refers to how a graph behaves as *x* tends toward infinity or negative infinity. For polynomial and rational functions, the *y*-values may get bigger and bigger positively or negatively as we move along the *x*-axis in either direction. Another possibility for rational functions is that the *y*-values may settle on a finite number *L*. If so, the line $y = L$ is called a **horizontal asymptote**. Note that a graph can cross or touch a horizontal asymptote. For a rational function, the leading terms in the numerator and denominator dictate the end behavior. We will verify this statement in Activity 4.3.

$$\lim_{x\to+\infty} f(x) = L$$



or

$$\lim_{x\to-\infty} f(x) = L$$



For a rational function, a **vertical asymptote** can be detected by analyzing the zeros of the numerator and denominator, but sometimes we want to know about behavior *near* an asymptote. We use **one-sided limits** to describe what happens to a graph as we approach an input from one side or the other:

Left-hand limit: $\lim_{x\to a^-} f(x)$ (*x* is approaching *a* **from the left**.)

Right-hand limit: $\lim_{x\to a^+} f(x)$ (*x* is approaching *a* **from the right**.)

Since we are typically interested in whether the one-sided limits are equal to each other, we use the **two-sided limit**, $\lim_{x\to a} f(x)$, to describe this behavior. A two-sided limit **exists** if the one-sided limits are equal and finite. In all other cases, we say that the limit **does not exist**. The following table will help us organize the different possibilities and notations.

Conditions on one-sided limits	What we say about the two-sided limit	How we write the two-sided limit
$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$	The limit **exists** and equals *L*.	$\lim_{x\to a} f(x) = L$
$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \pm\infty$	The limit **does not exist** but equals ±∞.	$\lim_{x\to a} f(x) = \pm\infty$
$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ DNE or $\lim_{x\to a^+} f(x)$ DNE (Here, DNE means the limit neither settles on a value nor grows without bound.)	The limit **does not exist**.	$\lim_{x\to a} f(x)$ DNE